Relativistic kinematics.

Note Title 4/14/2016

We have formulated the notion of 4-vector

(12.1) $$dx^i = (c\,dt,\ d\vec{r});$$

4-velocity: $$u^i = \frac{dx^i}{ds},$$

where $$ds = \sqrt{c^2dt^2 - dr^2} = c\,dt\sqrt{1-\frac{v^2}{c^2}}$$

"usual" 3d velocity

$$\Rightarrow u^0 = \frac{c\,dt}{c\,dt\sqrt{1-\dots}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

(12.2)

$$u^{1\dots3} = \frac{dx\dots}{c\,dt\sqrt{\dots}} = \frac{v_{x\dots}}{c\sqrt{1-v^2/c^2}}$$

$$u^i u_i = u^0u^0 - \sum_{i=1}^{3} u^{i2} = \frac{1}{1-\frac{v^2}{c^2}}\left[1 - \frac{v^2}{c^2}\right] = 1$$

To derive kinematic equations, we start with Lagrangian:

(12.3) $$L = -mc^2\sqrt{1-\frac{v^2}{c^2}}$$

Note that for $v \ll c$

$$L \simeq -mc^2\left(1-\frac{1}{2}\frac{v^2}{c^2}\right) = -mc^2 + \frac{mv^2}{2}$$

momentum

(12.4) $$\vec{p} = \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial v} = \frac{+mc^2\,2v}{c^2 2\sqrt{1-\frac{v^2}{c^2}}} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$p^2 = \frac{m^2v^2}{1-\frac{v^2}{c^2}} \Rightarrow p^2 - \frac{p^2v^2}{c^2} = m^2v^2$$

$$v^2 = \frac{p^2}{m^2+\frac{p^2}{c^2}}$$

Hamiltonian (energy) is defined by

$$\mathcal{E} = \bar{p}\bar{v} - L = \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + mc^2\sqrt{1-\frac{v^2}{c^2}} =$$

$v \ll c$

$$(12.5)\quad \mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \simeq mc^2 + \frac{mv^2}{2}$$

$$(12.6)\quad \mathcal{H} = \frac{mc^2}{\sqrt{1-\frac{p^2}{c^2m^2+p^2}}} = \frac{mc^2\sqrt{p^2+m^2c^2}}{mc} = c\sqrt{p^2+m^2c^2}$$

In the 4-vector form:

$$(12.7)\quad p^i = \left(\frac{\mathcal{E}}{c}, \vec{p}\right);$$ (=> transformation laws!)

4-vector of the force: $g^i = \frac{dp^i}{ds} = mc\frac{du^i}{ds}$

$$(12.8)\quad g = \left[\frac{\vec{f}\cdot\vec{v}}{c^2\sqrt{1-v^2/c^2}}, \frac{\vec{f}}{c\sqrt{1-v^2/c^2}}\right]$$

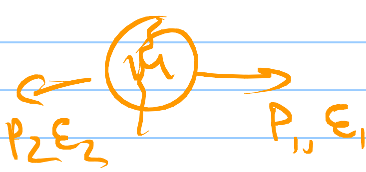
Recipe for particle decay:

1 Go to reference frame where particle is stationary (or center-of-mass system where $\vec{p}=0$)

2. Solve

$$Mc^2 = \mathcal{E}_1 + \mathcal{E}_2$$

$$\vec{p}_1 = -\vec{p}_2$$



$$\Rightarrow p_1^2 = p_2^2 \Leftrightarrow \mathcal{E}_1^2 - c^2m_1^2 = \mathcal{E}_2^2 - c^2m_2^2$$

3. If needed, transform back to original ref. frame

Electromagnetic potentials

(12.9) $$S = \int \left(-mc\, ds - \frac{e}{c} A_i\, dx^i\right)$$

where $$A^i = (\varphi, \vec{A})$$

scalar potential

vector potential

Note the units!

changing $ds \to dt$:

$$S = \int -mc^2 dt \sqrt{1-\frac{v^2}{c^2}} - \frac{e}{c}\left[\varphi \cdot c\,dt - \vec{A}\cdot d\vec{r}\right] =$$

$$= \int \left[-mc^2\sqrt{1-\frac{v^2}{c^2}} - e\cdot\varphi + \vec{A}\cdot\vec{v}\,\frac{e}{c}\right] dt$$

$$\Rightarrow L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\cdot\varphi + \vec{A}\cdot\vec{v}\,\frac{e}{c}$$

(12.10) $$\vec{P} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} + \vec{A}\frac{e}{c} = \vec{p} + \vec{A}\frac{e}{c}$$

generalized momentum

linear momentum

$$\mathcal{H} = \vec{v}\cdot\frac{\partial L}{\partial \vec{v}} - L = \frac{mv^2}{\sqrt{1-v^2/c^2}} + \cancel{\vec{A}\cdot\vec{v}\frac{e}{c}} + mc^2\sqrt{\cdots} + e\varphi \cancel{-\vec{A}\cdots}$$

$$= \frac{mc^2}{\sqrt{\cdots}} + e\varphi$$

comparing with 12.5, obtain

(12.11) $$\left(\frac{\mathcal{H}-e\varphi}{c}\right)^2 = \left(\frac{mc}{\sqrt{\cdots}}\right)^2 = m^2c^2 + \left(\vec{P} + \frac{e}{c}\vec{A}\right)^2$$

Consider motion of the charge in EM fields

Eq. of motions

(12.12) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = \frac{\partial L}{\partial \vec{r}} = \vec{\nabla} L = -e\vec{\nabla}\varphi + \frac{e}{c}\vec{\nabla}(\vec{A}\cdot\vec{v})$$

$$\vec{\nabla}(\vec{A}\cdot\vec{v}) = \cancel{(\vec{A}\cdot\vec{\nabla})\vec{v}} + (\vec{v}\cdot\vec{\nabla})\vec{A} + [\vec{v}\times\vec{\nabla}\times\vec{A}] + \cancel{[\vec{A}\times\vec{\nabla}\times\vec{v}]}$$

$$\frac{d}{dt}\left(\vec{p} + \frac{e}{c}\vec{A}\right) = -e\vec{\nabla}\varphi + \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A} + \frac{e}{c}\vec{v}\times(\vec{\nabla}\times\vec{A})$$

$$\frac{d}{dt}\vec{p} + \frac{e}{c}\frac{\partial\vec{A}}{\partial t} + \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A} = -e\vec{\nabla}\varphi + \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A} + \frac{e}{c}\vec{v}\times(\vec{\nabla}\times\vec{A})$$

(12.13) $$\frac{d}{dt}\vec{p} = e\underbrace{\left(-\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)}_{\vec{E}} + \frac{e}{c}\vec{v}\times\underbrace{(\vec{\nabla}\times\vec{A})}_{\vec{H}}$$

Note that

(12.14) $$\frac{d}{dt}E_{kin} = \frac{d}{dt}\frac{mc^2}{\sqrt{1-v^2/c^2}} = \vec{v}\cdot\frac{d\vec{p}}{dt} = e\vec{E}\cdot\vec{v}$$

Eqs of motion in covariant form :

(12.9) $$S = -\int\left(mc\,ds + \frac{e}{c}A_i\,dx^i\right)$$

Eq. of motion : $\delta S = 0$

$$ds^2 = dx_i\,dx^i$$

(12.15) $$\delta S = -\delta\int\left(\frac{mc\,dx_i\,dx^i}{ds} + \frac{e}{c}A_i\,dx^i\right) =$$ by parts

$$= -\int mc\,\frac{\boxed{dx_i}}{ds}d\delta x^i + \frac{e}{c}\delta A_i dx^i + \frac{e}{c}A_i d\delta x^i =$$

u_i

$$= -\cancelto{0}{\left(mc\,u_i + \frac{e}{c}A_i\right)\delta x^i\Big|_a^b} + \int mc\,du_i\,\delta x^i + \frac{e}{c}\,dA_i\,\delta x^i - \frac{e}{c}\,\delta A_i\,dx^i = 0$$

$$dA_i = \frac{\partial A_i}{\partial x^k}\,dx^k \quad ; \quad \delta A_i = \frac{\partial A_i}{\partial x^k}\,\delta x^k$$

$$= \delta S = \int mc\,du_i\,\delta x^i + \frac{e}{c}\frac{\partial A_i}{\partial x^k}\,dx^k\,\delta x^i - \frac{e}{c}\frac{\partial A_i}{\partial x^k}\,\delta x^k\,dx^i$$

$$= \int \left[mc\,du_k - \frac{e}{c}\left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}\right)\underbrace{dx^i}_{u^i ds}\right]\delta x^k = 0$$

(12.16)

$$\Rightarrow \quad mc\,\frac{du_k}{ds} = \frac{e}{c}\underbrace{\left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}\right)}_{F_{ki}\ -\ \text{em field tensor}} u^i$$

$$\Downarrow$$

$$mc\,\frac{du^i}{ds} = \frac{e}{c}\,F^{ik}\,u_k$$

Since $A^i = (\varphi, \vec{A})$, $A_i = (\varphi, -\vec{A})$

(12.17)

$$F_{ik} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{bmatrix} \quad ; \quad F^{ik} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{bmatrix}$$

Note that since $\vec{E}, \vec{H}$ form a 4-tensor they transform according to tensor rules:

From Lorentz transforms

(12.18)

$$E_x = F^{10} = \frac{\partial x^1}{\partial x'^i}\frac{\partial x^0}{\partial x'^j}F'^{ij} = \frac{\partial x^1}{\partial x'^1}\frac{\partial x^0}{\partial x'^0}F'^{10} + \frac{\partial x^1}{\partial x'^0}\frac{\partial x^0}{\partial x'^1}\underbrace{F'^{01}}_{=-F'^{10}} = \frac{1}{1-\frac{v^2}{c^2}}\left[1-\frac{v^2}{c^2}\right]F'^{10} = E_x'$$

$$E_y = F^{20} = \frac{\partial x^2}{\partial x'^2}\frac{\partial x^0}{\partial x'^0}F'^{20} + \frac{\partial x^2}{\partial x'^2}\frac{\partial x^0}{\partial x'^1}F'^{21} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[E_y' + \frac{v}{c}H_z'\right]$$

$$E_z = F^{30} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[E_z' - \frac{v}{c}H_y'\right]$$

$$H_x = F^{32} = H_x'$$

$$H_y = F^{13} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[F'^{13} + \frac{v}{c}F'^{03}\right] = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[H_y' - \frac{v}{c}E_z'\right]$$

$$H_z = F^{21} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[F'^{21} + \frac{v}{c}F'^{20}\right] = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[H_z' + \frac{v}{c}E_y'\right]$$

In the limit $\frac{v}{c} \ll 1$,

(12.18b)

$$\begin{cases} E_x = E_x', \quad E_y = E_y' + \frac{v}{c}H_z, \quad E_z = E_z' - \frac{v}{c}H_y' \\ H_x = H_x', \quad H_y = H_y' - \frac{v}{c}E_z, \quad H_z = H_z' + \frac{v}{c}E_y' \end{cases}$$

$$\Rightarrow \quad \vec{E} = \vec{E}' - \frac{1}{c}\left[\vec{v}\times\vec{H}'\right]; \quad \vec{H} = \vec{H}' + \frac{1}{c}\left[\vec{v}\times\vec{E}'\right]$$

Note that transformation of a tensor keeps some quantities constant:

$$(2.19)\begin{cases} F_{ik}F^{ik} = inv. \Rightarrow \vec{H}^2 - E^2 = inv = H'^2 - E'^2 \\ e^{iklm} F_{ik} F_{lm} = inv \Rightarrow \vec{E} \cdot \vec{H} = inv = \vec{E}' \cdot \vec{H}' \end{cases}$$