Note Title 4/12/2016

Special theory of relativity. 4-vectors

The main idea behind relativity is that all processes should be identical in all inertial frames. The second crucial point is that the information moves with certain max. speed.

Consider the event when a signal is emitted from pt. 1 and propagates to pt. 2 during interval $t_1 \to t_2$ in one inertial frame then:

(4.1) $$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-c^2(t_2-t_1)^2=0$$

in a different inertial reference frame,

$$(x_2'-x_1')+(y_2'-y_1')+(z_2'-z_1')-c^2(t_2'-t_1')=0$$

$\Rightarrow$ differentiate $ds^2 = c^2dt^2 - d\vec{r}\cdot d\vec{r}$

remains the same. ($=0$ in both systems)

In general, we see that

(11.2) $$ds^2 = a\, ds'^2$$

A constant a may be a function of relative velocity V'; Consider now another inertial reference frame k''. then:

$$ds^2 = a(V')ds'^2 = a(V'')ds''^2$$

but $ds'^2 = a(v'''') ds''^2$

$\Rightarrow a(v'^{\to}'') = \dfrac{a(v'')}{a(v')}$;

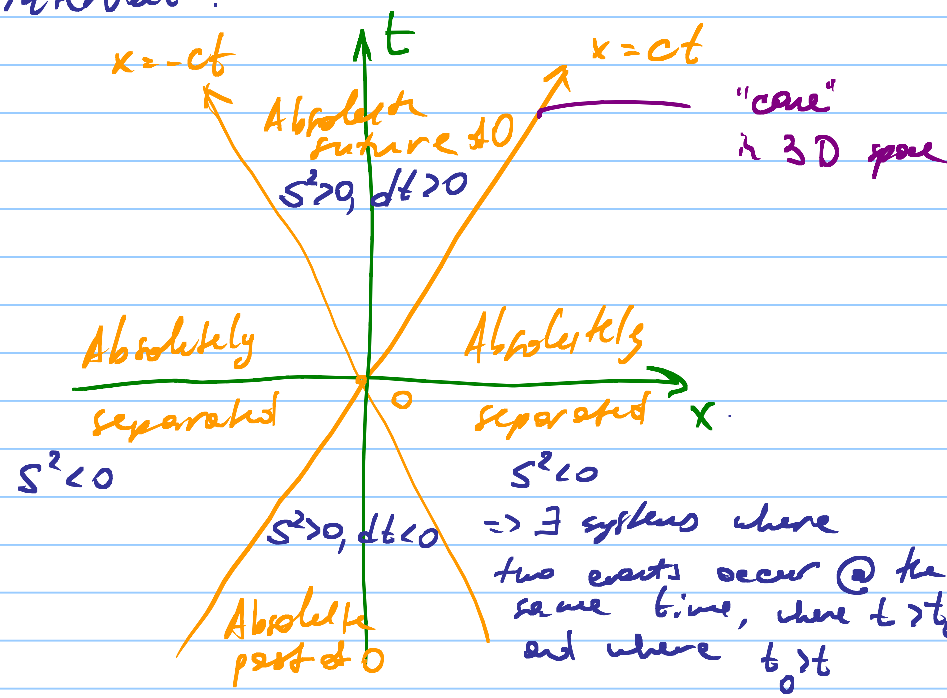
however since relative velocity is vector quantity, the only scalar meaningful relationship is therefore

$a(v) = 1$

we therefore see that the interval remains invariant in Special theory of relativity.

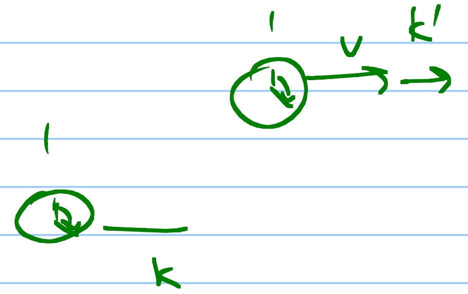
(1.3) $s^2 = c^2dt^2 - dr^2 = c^2dt'^2 - dr'^2$

$\Rightarrow$ all events can be grouped according to the interval:



$\Rightarrow \exists$ systems where two events occur @ the same time, where $t > t_0$ and where $t_0 > t$

Coordinate transformations



Assume that we observe a clock moving with respect to our rest frame @ speed V

then the interval

(11.4) $$ds^2 = \underbrace{c^2dt^2 - dx^2}_{\text{our rest frame, moving clock}} = \underbrace{c^2dt'^2}_{\text{clock's rest frame (proper time)}}$$

(11.5) $$dt' = dt\sqrt{1-\frac{dx^2}{c^2dt^2}} = dt\sqrt{1-\frac{V^2}{c^2}}$$

Note: $dt' < dt \Rightarrow$ moving clocks appear slower than clocks @ rest

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The general coordinate transformation must leave interval unchanged $\Rightarrow$ transformation must be a rotation.

For simplicity, choose the relative velocity of the two systems as $\hat{x}$ $\Rightarrow$

(11.6) $$\begin{cases} x = x'\cosh\psi + ct'\sinh\psi \\ ct = x'\sinh\psi + ct'\cosh\psi \end{cases}$$

the <u>origin</u> of K' $(x'=0)$ moves as:

$$\begin{cases} x = ct'\sinh\psi \\ ct = ct'\cosh\psi \end{cases} \Rightarrow \frac{x}{ct} = \frac{V}{c} = \tanh\psi$$

(11.7) $\cosh^2\psi - \sinh^2\psi = 1 \Rightarrow \cosh^2\psi = \dfrac{1}{1-\tanh^2\psi}$

$$\sinh^2\psi = \frac{\tanh^2\psi}{1-\tanh^2\psi}$$

(11.8)
$$\Rightarrow \begin{cases} x = \dfrac{x'}{\sqrt{1-\frac{v^2}{c^2}}} + ct'\dfrac{V}{c}\dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \dfrac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}} \\ t = \dfrac{x'V}{c\sqrt{1-v^2/c^2}} + \dfrac{ct'}{\sqrt{1-v^2/c^2}} = \dfrac{t'+x'\frac{V}{c^2}}{\sqrt{1-v^2/c^2}} \\ y = y' \ ; \ z = z' \end{cases}$$

(11.9) $\Rightarrow \Delta x = \dfrac{\Delta x'}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \Delta x' = \Delta x\sqrt{1-\frac{v^2}{c^2}} < \Delta x$

proper length (length in object's rest frame)

From (11.8), obtain:

$$dx = \frac{dx'+Vdt'}{\sqrt{1-\frac{v^2}{c^2}}} \ ; \ dt = \frac{dt'+dx'\frac{V}{c^2}}{\sqrt{1-v^2/c^2}}$$

$$dy = dy' \ ; \ dz = dz'$$

Introducing $\vec{v} = \dfrac{d\vec{r}}{dt}$; $\vec{v}' = \dfrac{d\vec{r}'}{dt'}$,

obtain:

(11.10)
$$\begin{cases} v_x = \dfrac{dx}{dt} = \dfrac{dx'+Vdt'}{dt'+dx'\frac{V}{c^2}} = \dfrac{v_x'+V}{1+\frac{v_x'V}{c^2}} \\ v_y = \dfrac{dy}{dt} = \dfrac{dy'}{dt'+\frac{dx'V}{c^2}}\sqrt{1-V^2/c^2} = \dfrac{v_y'}{1+\frac{v_x'V}{c^2}}\sqrt{1-\frac{V^2}{c^2}} \\ v_z = \dfrac{v_z'}{1+v_x'V/c^2}\sqrt{1-\frac{V^2}{c^2}} \end{cases}$$

when $V \ll c$,

$$v_x = (v_x' + V)\left(1 - \frac{v_x' V}{c^2}\right) = v_x' + V - \frac{v_x'^2}{c^2} V - \cancel{\frac{V^2}{c^2} v_x'}^{\,0}$$

$$\simeq v_x' + V\left(1 - \frac{v_x'^2}{c^2}\right)$$

$$v_y = v_y'\left(1 - \frac{v_x' V}{c^2} + \ldots\right)\left(1 - \frac{1}{2}\frac{V^2}{c^2} \ldots\right) \simeq$$

$$\simeq v_y' - v_x' v_y' \frac{V}{c^2}$$

$$v_z \simeq v_z' - v_x' v_z' \frac{V}{c^2}$$

$$(11,4) \quad \boxed{\vec{v} = \vec{v}\,' + \vec{V} - \frac{1}{c^2}(\bar{v}' \cdot \bar{V})\,\vec{v}\,'}$$

The transformations above can be considered as transformations of vectors in 4D space with metrics:

metric tensor

$$g^{ik} = g_{ik} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$$

$$dx^0 = c\,dt, \quad dx^1 \ldots dx^3 = d\vec{r}$$

contravariant components of a 4-vector

$$ds^2 = dx^i dx_i = g_{ik} dx^i dx^k$$

covariant components of a 4-vecs

Note: $dx_0 = dx^0; \quad dx_1 \ldots dx_3 = -dx^1 \ldots dx^3$

Useful 4-tensors. $\delta^i_k = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$

$$e^{ijkl} = \begin{cases} 1, & ijkl = 0123 \\ \pm 1, & \text{depending on permutation of } ijkl \\ 0, & \text{if either two indices are the same} \end{cases}$$

$$e_{ijkl} = \begin{cases} -1, & ijkl = 0123 \\ \pm 1, & \ldots \\ 0, & \ldots \end{cases}$$

Covariant derivative:

$$\frac{\partial \varphi}{\partial x^i} = \left(\frac{1}{c} \frac{\partial \varphi}{\partial t}, \vec{\nabla} \varphi \right);$$

$$d\varphi = \frac{\partial \varphi}{\partial x^i} dx^i \quad \text{- scalar}$$

contravariant 4-vector

covariant - 4-vector

Transformations of vectors & tensors

As in any orthogonal coordinate system

$$V^i = \frac{\partial x^i}{\partial x'^k} V'^k$$

for Lorentz transformation;

$$V^0 = \frac{V^{0'} + \frac{v}{c} V^{1'}}{\sqrt{1 - \frac{v^2}{c^2}}}; \quad V^1 = \frac{V^{1'} + \frac{v}{c} V^{0'}}{\sqrt{1 - \frac{v^2}{c^2}}}; \quad V^2 = V^{2'}; \quad V^3 = V^{3'}$$

$$A^{ij} = \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} A'^{kl}$$

$$\Rightarrow A^{00} = \frac{\partial x^0}{\partial x'^0} \frac{\partial x^0}{\partial x'^0} A'^{00} + \frac{\partial x^0}{\partial x'^1} \frac{\partial x^0}{\partial x'^1} A'^{11} + \frac{\partial x^0}{\partial x'^0} \frac{\partial x^0}{\partial x'^1} \left(A'^{01} + A'^{10} \right)$$

$$= \frac{1}{1 - \frac{v^2}{c^2}} \left[A'^{00} + \frac{v^2}{c^2} A'^{11} + \frac{v}{c} \left(A'^{01} + A'^{10} \right) \right]$$

$$A^{11} = \frac{1}{1-\frac{v^2}{c^2}}\left[A'^{11} + \frac{v^2}{c^2}A'^{00} + \frac{v}{c}\left(A'^{01} + A'^{10}\right)\right]$$

$$A^{10} = \frac{\partial x^1}{\partial x'^1}\frac{\partial x^0}{\partial x'^1}A'^{11} + \frac{\partial x^1}{\partial x'^1}\frac{\partial x^0}{\partial x'^0}A'^{10} + \frac{\partial x^1}{\partial x'^0}\frac{\partial x^0}{\partial x'^1}A'^{01} +$$

$$+ \frac{\partial x^1}{\partial x'^0}\frac{\partial x^0}{\partial x'^0}A'^{00} = \frac{1}{1-\frac{v^2}{c^2}}\left(\frac{v}{c}\left(A'^{11} + A'^{00}\right) + A'^{10} + \frac{v^2}{c^2}A'^{01}\right)$$

$$A^{02} = \frac{\partial x^0}{\partial x'^1}\frac{\partial x^2}{\partial x'^2}A'^{12} + \frac{\partial x^0}{\partial x'^0}\frac{\partial x^2}{\partial x'^2}A'^{02} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{v}{c}A'^{12} + A'^{02}\right)$$

$$A^{22} = A'^{22}$$

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