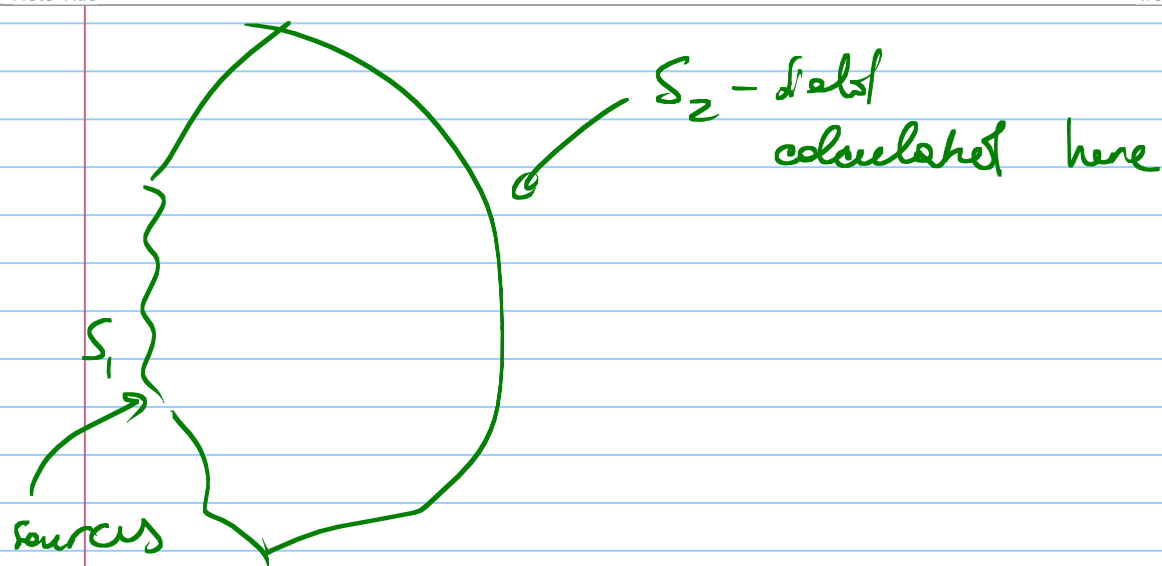
Diffraction

Note Title 4/8/2016



Consider first Green's identity:

(10.1) $\int dV \quad \vec{\nabla}\cdot(\varphi\vec{\nabla}\psi) = \vec{\nabla}\varphi\cdot\vec{\nabla}\psi + \varphi\nabla^2\psi$

$\vec{\nabla}\cdot(\psi\vec{\nabla}\varphi) = \vec{\nabla}\psi\cdot\vec{\nabla}\varphi + \psi\nabla^2\varphi$

(10.2)
$$\int d^3v\,(\varphi\nabla^2\psi - \psi\nabla^2\varphi) = \oint(\varphi\vec{\nabla}\psi - \psi\vec{\nabla}\varphi)\cdot d\vec{a}$$
$$= \oint\left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right)da$$

Here we consider scalar diffraction theory: Consider a component of the field of an EM wave; in homogeneous space, obtain

(10.3)
$$(\nabla^2 + k^2)\psi = 0$$

When sources are considered, it is convenient to use Green function:

(10.4)
$$(\nabla^2 + k^2)\,G(\bar{r},\bar{r}') = -\delta(\bar{r}-\bar{r}')$$

Substituting $\varphi = G$ into (10.2)

$$\oint_S\left(G\frac{\partial\psi}{\partial n} - \psi\frac{\partial G}{\partial n'}\right)da = \int d^3v\,(G\nabla^2\psi - \psi\nabla^2 G) =$$

$$= \int d^3r \left(G k^2 \psi + \psi k^2 G + \delta(\bar r - \bar r')\psi \right) =$$

$$= \psi(\bar r)$$

(10.5) $$\Rightarrow \psi(\bar r) = \oint \left[\frac{e^{ikR}}{4\pi R} \frac{\partial \psi}{\partial n} - \psi \frac{\partial}{\partial n'} \frac{e^{ikR}}{4\pi R} \right] d^2\vec a :$$

$$R = |r - r'|; \quad \frac{\partial}{\partial n} \frac{e^{ikR}}{R} = -\frac{e^{ikR}}{R^2} + ik \frac{e^{ikR}}{R} =$$

$$= ik \frac{e^{ikR}}{R} \left(1 + \frac{i}{kR}\right)$$

$\vec n$ - outer normal

$$= \frac{1}{4\pi} \oint \frac{e^{ikR}}{R} \left[\frac{\partial \psi(r')}{\partial n} - ik\left(1 + \frac{i}{kR}\right)\psi(r') \right] d^2a$$

$$= -\frac{1}{4\pi} \oint \frac{e^{ikR}}{R} \vec n' \cdot \left[\vec\nabla' \psi(r') + ik\left(1 + \frac{i}{kR}\right)\frac{\vec R}{R}\psi(r') \right] d^2\vec a$$

inner normal

(10.6) $$= -\frac{1}{4\pi} \int_{S_1} \frac{e^{ikR}}{R} \vec n' \cdot \Big[\underbrace{\nabla' \psi(r')}_{\text{dominates Dirichlet B.C}} + \underbrace{ik\left(1 + \frac{i}{kR}\right)\frac{\vec R}{R}\psi(r')}_{\text{dominates Neumann b.C}} \Big] d^2a$$

According to Kirchhoff's approximation

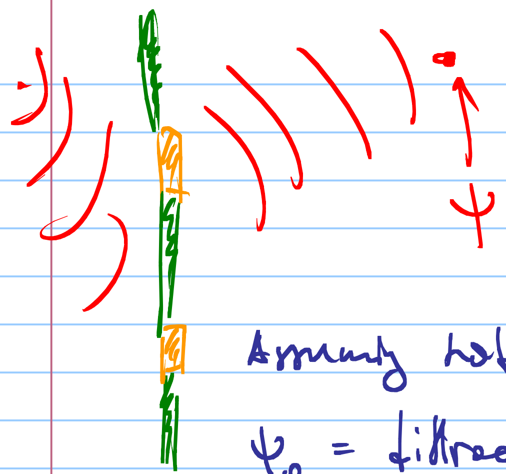
$G|_S = 0$

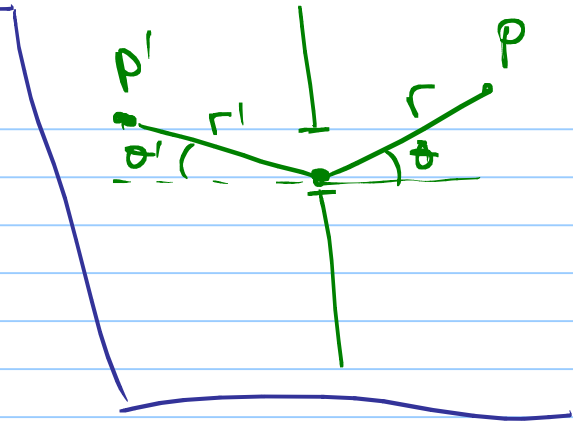
$$\begin{cases} \psi, \left.\frac{\partial \psi}{\partial n}\right|_{\text{opening}} = \psi, \left.\frac{\partial \psi}{\partial n}\right|_{\text{incident wave}} \\ \psi, \left.\frac{\partial \psi}{\partial n}\right|_{\text{screen}} = 0 \end{cases}$$

$$\frac{\partial G}{\partial n} = 0$$

For a diffraction due to a point source behind a screen:

(10.7) $$\psi(P) = \frac{k}{2\pi i} \int_{\text{appertures}} \frac{e^{ikr}}{r} \frac{e^{ikr'}}{r'} \frac{1}{2}(\cos\theta' + \cos\theta)\, da'$$





Assuming that

ψ_a = diffracted field with screen a

ψ_b = diffracted field with complementary screen

$\Rightarrow \psi_a + \psi_b = \psi \leftarrow$ field in the absence of screen

(Babinet principle)

Same problem can be considered in the Fourier domain:

Field @ the screen

(10.8) $$\left\{ \begin{array}{l} u(x,y) = \frac{1}{4\pi^2} \iint U_k(k_x,k_y) e^{ik\cdot r}\, dk_x\, dk_y \\ \text{where } U_k = \iint u(x,y)\, e^{-ik\cdot r}\, dx\, dy \end{array} \right.$$

Consider the limit when the incident light is a plane wave and when the diffraction is weak; in this case the deviation from the plane wave is given by the wavevector

(10.9) $$\bar{q} = k - \bar{k}_0 \ll \frac{\omega}{c} \simeq \frac{\omega}{c} \cdot \bar{\theta}$$

the intensity is given by

(10.10) $$\iint |u_0^2|\, dx\, dy = \iint |u_k(k)|^2 \frac{d^2 q_\perp}{4\pi^2}$$

the relative intensity diffracted into solid angle

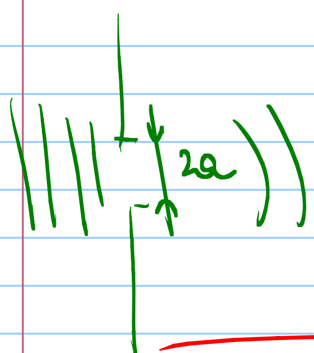
$d\Omega = d\theta_x\, d\theta_y$:

(10.11) $$\frac{|u_k|^2}{u_0^2} \frac{dq_x\, dq_y}{4\pi^2} = \left(\frac{\omega}{2\pi}\right)^2 \left|\frac{u_k}{u_0}\right|^2 d\Omega$$

In fourier domain Babinet principle means

$$u_k^a + u_k^b = 0 \,, \quad q \neq 0 \quad (\text{scattered light})$$

$$\Rightarrow u_k^a = -u_k^b \Rightarrow |u_k^a|^2 = |u_k^b|^2$$



$$u_k = u_0 \int_{-a}^{a} e^{-ik_x x} dx = 2u_0 \frac{\sin k_x a}{k_x}$$

the intensity of diffracted light:

(10.12) $$\boxed{dI(\theta) \sim \left|\frac{u_k}{u_0}\right|^2 \frac{dq}{2\pi} \cdot \frac{I_0}{2a} = \frac{I_0}{2a} \frac{\sin^2\left(\frac{\omega}{c} a \theta\right)}{\frac{\omega}{c}\theta^2} d\theta}$$

Let's now look @ N slits, with a each distance d from each other. Then

$$u_k = u_0 \sum_{m=0}^{N-1} \int_{-a}^{a} e^{-ik_x(x-md)} dx =$$

$$= \sum_m \frac{u_0}{-ik_x} \left[e^{-ik_x(a-md)} - e^{ik_x(a+md)} \right] =$$

$$= \frac{u_0}{-ik_x}\left[e^{-ik_x a}\sum_m e^{ik_x md} - e^{ik_x a}\sum_m e^{ik_x md}\right] =$$

$$= \frac{u_0}{-ik_x}\left(e^{-ik_x a} - e^{ik_x a}\right)\frac{e^{ik_x Nd} - 1}{1 - e^{ik_x d}} =$$

$$= \frac{2u_0}{k_x}\sin k_x a \; \frac{e^{ik_x \frac{Nd}{2}}}{e^{ik_x d/2}} \; \frac{e^{ik_x Nd/2} - e^{-ik_x Nd/2}}{e^{-ik_x d/2} - e^{ik_x d/2}} =$$

$$= 2u_0 \frac{\sin k_x \frac{Nd}{2}}{\sin k_x \frac{d}{2}} \; \frac{\sin k_x a}{k_x} \; e^{ik_x (N-1)\frac{d}{2}}$$

$$dI \propto \left|\frac{u_k}{u_0}\right|^2 \frac{dq}{2\pi} \frac{I_0}{2Na} = \frac{I_0 k\, d\theta}{\pi N a} \; \frac{\sin^2 \frac{kNd\theta}{2}}{\sin^2 \frac{kd\theta}{2}} \; \frac{(\sin \theta k a)^2}{k^2\theta^2} =$$

(10.13) $$\boxed{= \frac{I_0}{\pi N a k}\left[\frac{\sin \frac{kNd\theta}{2}}{\sin \frac{kd\theta}{2}} \; \frac{\sin k a \theta}{\theta}\right]^2 d\theta}$$