Scattering

Note Title 3/25/2016

Consider a (slightly) inhomogeneous medium

In the process of scattering incident plane wave is gradually converted into scattered wave

(9.1) $$D_i^{sc} = \varepsilon^s E_i^{sc} + \alpha_{ik} E_k^{inc} \quad ;$$

scattering amplitude

Assume that scattering is primarily "dielectric"

(9.1a) $$\Rightarrow \quad B^{sc} = \mu_0 H^{sc}$$

Maxwell eqn for scattered fields:

(9.2) $$\begin{cases} \bar\nabla \times \bar H^{sc} = \dfrac{\partial D^{sc}}{\partial t} \; ; \quad \bar\nabla \times \bar E^{sc} = -\dfrac{\partial B^{sc}}{\partial t} \\ \bar\nabla \times \bar\nabla \times \bar E^{sc} = -\dfrac{\partial}{\partial t} \bar\nabla \times \bar B^{sc} = -\dfrac{\partial}{\partial t}\mu_0 \dfrac{\partial D^{sc}}{\partial t} \end{cases}$$

$$\bar\nabla \times \bar\nabla \times \bar E^{sc} = \omega^2 \mu_0 \bar D^{sc}$$

$$\bar E^{sc} = \frac{1}{\varepsilon^{sc}} D^{sc} - \frac{1}{\varepsilon^{sc}} \hat\alpha \bar E^{inc} \quad \left(\varepsilon^{sc} \neq \varepsilon^{sc}(\bar r)\right)$$

(9.3) $$\nabla \times \nabla \times \left(\frac{\bar D^{sc}}{\varepsilon^{sc}} - \frac{1}{\varepsilon^{sc}} \hat\alpha \vec E^{inc}\right) = \omega^2 \mu_0 \bar D^{sc} \quad \left(\vec\nabla \cdot \bar D^{sc} = 0\right)$$

$$-\nabla^2 D^{sc} - \bar\nabla \times \bar\nabla \times (\alpha E^{inc}) = \omega^2 \varepsilon^{sc} \mu_0 D^{sc}$$

(9.4) $$\boxed{(\nabla^2 + k^2) \bar D^{sc} = -\bar\nabla \times \bar\nabla (\hat\alpha \bar E^{inc}) = \underbrace{-\bar\nabla \times \bar\nabla \times \left(\bar D^{sc} - \varepsilon^s \bar E^{sc}\right)}_{\text{"source" term}}}$$

solution in absence of source term

(9.5) $$\boxed{\bar D^{sc} = \vec D^{(0)} + \frac{1}{4\pi} \int d^3 r' \, \frac{e^{ik|r-r'|}}{|r-r'|} \left[\nabla' \times \nabla' \times \left(\bar D^{sc} - \varepsilon^s \bar E^{sc}\right)\right]}$$

In the far field:

(9.6)
$$\vec{D} \simeq \vec{D}^0 + \vec{A}_{sc}\,\frac{e^{ikr}}{r},$$

(9.7)
$$\vec{A}_{sc} = \frac{1}{4\pi}\int d^3r'\, e^{ik\hat{r}\cdot\vec{r}'}\left[\nabla'\times\nabla'\times\left(D^{sc} - \epsilon^s E^s\right)\right] =$$
$$= \frac{k^2}{4\pi}\int d^3r'\, e^{ik\hat{r}\cdot\vec{r}'}\left[\hat{r}\times\left(D^{sc} - \epsilon^s E^s\right)\times\hat{r}\right]$$

yielding differential scat. cross section:

(9.8)
$$\frac{d\sigma}{d\Omega} = \frac{|\vec{\epsilon}^*\cdot\vec{A}_{sc}|^2}{|D^0|^2}$$

Born approximation:

(9.9)
$$\bar{D}^{sc} = \left(\epsilon^s + \delta\epsilon\right)\bar{E}^{sc}$$
$$\bar{D}^{sc} - \epsilon^s\bar{E}^s = \delta\epsilon\,\bar{E}^{sc} \simeq \frac{\delta\epsilon}{\epsilon^{sc}}\,\vec{D}^0$$
$$\Rightarrow \vec{A}_{sc} = \frac{k^2}{4\pi}\int d^3r'\, e^{ik\hat{r}\cdot\vec{r}'}\,\frac{\delta\epsilon(r')}{\epsilon^{sc}}\overbrace{\left[\hat{r}\times\vec{D}_0\times\hat{r}\right]}^{\vec{D}_\perp}$$
$$\vec{\epsilon}^*\cdot\vec{A}_{sc} = \frac{k^2}{4\pi}\left(\vec{\epsilon}^*\cdot\vec{\epsilon}\right)|D_0|\int d^3\bar{r}'\, e^{ik\hat{r}\cdot\vec{r}'}\,\frac{\delta\epsilon(r')}{\epsilon^{sc}}$$

in the limit $kr \ll 1$,
$$\int d^3\bar{r}'\, e^{ik\hat{r}\cdot\vec{r}'}\,\frac{\delta\epsilon(r')}{\epsilon^{sc}} \simeq \frac{\delta\epsilon}{\epsilon^{sc}}\cdot\frac{4\pi}{3}a^3$$

radius of "imaginary" sphere of scatterers

$$\Rightarrow \left|\frac{\bar{\epsilon}^*\cdot\vec{A}_{sc}}{D^0}\right| = \frac{k^2 a^3}{3}\,\frac{\delta\epsilon}{\epsilon^{sc}}\left(\vec{\epsilon}^*\cdot\vec{\epsilon}\right)$$

(9.10)
$$\boxed{\frac{d\sigma}{d\Omega} = k^4 a^6\left|\frac{\delta\epsilon}{3\epsilon^{sc}}\right|^2\left|\bar{\epsilon}^*\cdot\bar{\epsilon}\right|^2}$$

Consider an array of point scatterers (dilute gas)

$$\Rightarrow \quad \epsilon(\bar{r}') = \epsilon_0 \left[1 + \sum \gamma_{mol}\, \delta(\bar{r}-\bar{r}')\right] = \epsilon_0 \left[1 + N\gamma_{mol}\right]$$

(9.11) $$\frac{\vec{\epsilon}^{*} \cdot \vec{A}_{sc}}{|D_0|} = \frac{k^2}{4\pi}\epsilon_0 \int d^3 r'\, e^{ik\hat{r}\cdot r'}\, \frac{\sum \gamma_{mol}\, \delta(\bar{r}-\bar{r}')}{\epsilon^{sc}} (\bar{\varepsilon}\cdot\bar{\varepsilon}^{*})$$

$$= \frac{k^2}{4\pi}\epsilon_0 \frac{N\gamma_{mol}}{\epsilon_{sc}} \left|\varepsilon \cdot \varepsilon^{*}\right|$$

$\sim \epsilon_0$

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2} \left(N\gamma_{mol}\right)^2 \left|\bar{\varepsilon}\cdot\bar{\varepsilon}^{*}\right|$$

(9.12) $$\underbrace{\sigma_{tot}}_{\text{per molecule}} = \frac{8\pi}{3} \frac{k^4}{16\pi^2} \frac{(\epsilon_r - 1)^2}{N^2} = \frac{k^4}{6\pi} \frac{(\epsilon_r - 1)^2}{N^2}$$