Emission

Note Title 3/15/2016

Emission is essentially an eigen response of a system to a combination of oscillatory charges & currents:

$$j_f,\ \rho_f \propto e^{-i\omega t}$$

In the simplest form, consider point-charge or currents:

(7.1) $$j_f,\ \rho_f = \left(j_0/\rho_0\right)\delta(r)\, e^{-i\omega t}$$

① Emission in free space:

We begin by re-writing potential formulation

(7.2) $$\begin{cases} \vec{B} = \bar{\nabla}\times\bar{A} \\ \vec{E} = -\vec{\nabla}\varphi - \dfrac{\partial\bar{A}}{\partial t} \end{cases}$$

Maxwell's eqs are now reduced to:

$$\mathrm{div}\,\vec{E} = \frac{\rho}{\epsilon_0} \rightarrow -\nabla^2\varphi - \frac{\partial}{\partial t}\vec{\nabla}\cdot\bar{A} = \frac{\rho}{\epsilon_0}$$

$$\bar{\nabla}\times\bar{H} = \epsilon_0\frac{\partial\bar{E}}{\partial t} + \bar{j} \rightarrow \bar{\nabla}\times\bar{\nabla}\times\bar{A} = \frac{1}{c^2}\left(-\bar{\nabla}\frac{\partial\varphi}{\partial t} - \frac{\partial^2 A}{\partial t^2}\right) + \mu_0\vec{j}$$

$$\vec{\nabla}\left(\bar{\nabla}\cdot\bar{A}\right) - \nabla^2\vec{A} = -\frac{1}{c^2}\vec{\nabla}\frac{\partial\varphi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \mu_0\vec{j}$$

(7.3) $$\boxed{\begin{aligned} &\nabla^2\varphi + \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\bar{A}\right) = -\frac{\rho}{\epsilon_0} \\ &\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \vec{\nabla}\left(\bar{\nabla}\cdot\bar{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right) = -\mu_0\vec{j} \end{aligned}}$$

Note that the fields are unchanged when

(7.4) $$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda;\quad \varphi \rightarrow \varphi - \frac{\partial\Lambda}{\partial t}$$ (Gauge transform)

(7.5) Lorentz Gauge: $$\bar{\nabla}\cdot\bar{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$$

⇒ Max. Eqs:

(25b)
$$\begin{cases} \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = -\frac{\rho}{\varepsilon_0} \\ \left(\nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{j} \end{cases}$$

Consider now the Green's function of the wave eqn:

(26)
$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(r, r', t, t') = -4\pi\,\delta(r-r')\,\delta(t-t')$$

looking for $G(r, r', t, t')$ in spectral representation

(27)
$$\begin{cases} G(r, r', t, t') = \frac{1}{2\pi}\int G_\omega(r, r', \omega, t')\, e^{-i\omega t}\, d\omega \\ \delta(t-t') = \frac{1}{2\pi}\int e^{-i\omega(t-t')}\, d\omega \end{cases}$$

$$\int \left(\nabla^2 + \frac{\omega^2}{c^2}\right) G_\omega(r, r', \omega, t')\, e^{-i\omega t}\, d\omega = -\int 4\pi\,\delta(r-r')\, e^{-i\omega(t-t')}\, d\omega$$

$$= \int d\omega\, e^{-i\omega t} \left[\left(\nabla^2 + \frac{\omega^2}{c^2}\right) G_\omega(r, r', \omega, t')\, e^{i\omega t'} + 4\pi\,\delta(r-r')\right] \Rightarrow$$

$$G_\omega e^{i\omega t'} = \frac{e^{\pm i\frac{\omega}{c}|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}$$

$$G(r, r', t, t') = \frac{1}{2\pi}\int \frac{e^{\pm i\frac{\omega}{c}|\bar{r}-\bar{r}'| - i\omega(t-t')}}{|r-r'|}\, d\omega =$$

(28)
$$= \frac{1}{2\pi|r-r'|}\int e^{-i\omega\left(t-t' \mp \frac{|r-r'|}{c}\right)}\, d\omega = \frac{\delta\left(t-t' \mp \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|r-r'|}$$

$\Rightarrow$ Solutions to (7.5b):

(7.9) $$\bar{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\bar{j}(r',t')}{|r-r'|}\,\delta\left(t-t'-\frac{|r-r'|}{c}\right)d^3\vec{r}'\,dt'$$

For monochromatic radiations

(7.10) $$j(\vec{r},t) = j(\vec{r})e^{-i\omega t}, \quad t' = t - \frac{|r-r'|}{c}$$

$$\bar{A}(r,t) = \frac{\mu_0}{4\pi}\int \frac{\bar{j}(\bar{r}')e^{-i\omega t'}}{|r-r'|}\,\underbrace{\delta\left(t-t'-\frac{|r-r'|}{c}\right)}_{t' = t - \frac{|r-r'|}{c}}d^3r'\,dt' =$$

$$= \frac{\mu_0}{4\pi}\int d^3r'\,\frac{j(r')e^{-i\omega\left[t-\frac{|r-r'|}{c}\right]}}{|r-r'|} =$$

(7.10) $$\boxed{\bar{A}(r,t) = \frac{\mu_0}{4\pi}e^{-i\omega t}\int d^3r'\,\frac{\vec{j}(r')e^{i\frac{\omega}{c}|r-r'|}}{|r-r'|}}$$

(field of the retarded potential)

with

(7.10a) $$\bar{H} = \frac{1}{\mu_0}\bar{\nabla}\times\bar{A}\ ; \quad \vec{E} = i\frac{1}{\omega\epsilon_0}\nabla\times\bar{H} = \boxed{i\frac{c}{\omega}\sqrt{\frac{\mu_0}{\epsilon_0}}}\,\bar{\nabla}\times\bar{H}$$

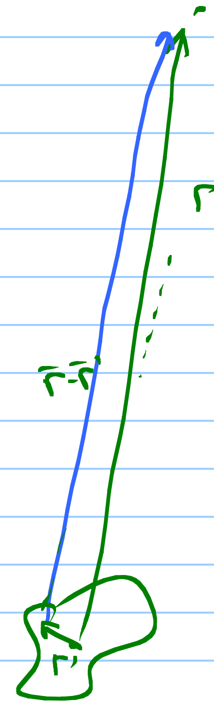
$i\frac{Z_0}{k}$

Fields of a localized source (size $d \ll \lambda$)

① near-field: $\boxed{d \ll r \ll \lambda}$

$$\Rightarrow e^{i\frac{\omega}{c}|r-r'|} = e^{2\pi i \frac{|r-r'|}{\lambda}} \simeq 1$$

(7.11) $$A \simeq \frac{\mu_0}{4\pi} e^{-i\omega t} \int d^3r' \, \frac{j(r')}{|r-r'|}$$



② far-field $\boxed{d \ll \lambda \ll r}$

$$A = \frac{\mu_0 e^{-i\omega t}}{4\pi} \int d^3r' \, \frac{j(\vec r')\, e^{i\frac{\omega}{c}|r-r'|}}{|r-r'|}$$

(7.12) $$|\vec r - \vec r'| = \sqrt{r^2 + r'^2 - 2\, r\cdot r'} \simeq$$

$$\simeq \vec r - \hat r \cdot \vec r' \,, \quad \hat r = \frac{\vec r}{r}$$

$$e^{i\frac{\omega}{c}|r-r'|} \simeq \exp\left(i\frac{\omega}{c}(r - \hat r\cdot\vec r')\right) \simeq$$

$$\simeq e^{i\frac{\omega}{c}r} + i\frac{\omega}{c}\exp\left(i\frac{\omega}{c}r\right)(-\hat r\cdot\vec r') - \frac{\omega^2}{c^2}\exp\left(i\frac{\omega}{c}r\right)(\hat r\cdot\vec r')^2 + \dots$$

$$\frac{1}{|r-r'|} \simeq \frac{1}{r - \hat r\cdot\vec r'} \simeq \frac{1}{r} + \frac{\hat r\cdot\vec r'}{r^2}$$

$$A \simeq \frac{\mu_0}{4\pi r} e^{-i\omega t + i\frac{\omega}{c}r} \int d^3r' \, \vec j(\vec r') \left[1 - i\frac{\omega}{c}\hat r\cdot\vec r'\right]\left[1 + \frac{\hat r\cdot\vec r'}{r}\right] \simeq$$

(7.13) $$\boxed{A \simeq \frac{\mu_0}{4\pi} \frac{e^{-i\omega t + ikr}}{r} \int d^3r' \, j(\vec r') \left[\underbrace{1}_{\text{dipole}} + \underbrace{\frac{\vec r\cdot\vec r'}{r^2} - i\vec k\cdot\vec r'}_{\text{quadrupole + magnetic dipole}}\right]}$$

leading term:

$$\int \underbrace{\vec j(r')}_{u}\,\underbrace{dr'}_{dv} = -\int \vec r'(\vec\nabla'\cdot\vec j)\,d^3r' = -i\omega\int \rho(r')\,\vec r'\,d^3r' =$$

$$\vec\nabla\cdot\vec j = -\frac{\partial\rho}{\partial t} = i\omega\rho \qquad\qquad = -i\omega\vec p$$

dipole moment

(7.14) $$\vec A = -\frac{i\omega\mu_0}{4\pi}\,\frac{e^{-i\omega t+ikr}}{r}\,\vec p$$

$$\vec H = \frac{\vec\nabla\times\vec A}{\mu_0} = -\frac{i\omega}{4\pi}\,\frac{e^{-i\omega t+ikr}}{r}\left(ik\frac{\times\vec p}{r} - \frac{\vec r\times\vec p}{r^2}\right)$$

$$\frac{\partial}{\partial x_i}\frac{e^{ikr}}{r} = \frac{1}{r}\left(e^{ikr}\cdot ik\cdot - \frac{e^{ikr}}{r}\right)\frac{\partial r}{\partial x_i} = \frac{e^{ikr}}{r}\left(ik-\frac{1}{r}\right)\frac{x_i}{r}$$

$$= +\frac{\omega k\cancel{\mu_0}}{4\pi}\,\frac{e^{-i\omega t+ikr}}{r}\,(\vec n\times\vec p)\left(1-\frac{1}{ikr}\right) \simeq$$

(7.15) $$H \simeq \begin{cases} \dfrac{k^2c}{4\pi}\,\dfrac{e^{-i\omega t+ik\cdot r}}{r}\,(n\times p), & r\gg\lambda \\[2ex] \dfrac{\omega}{4\pi}\,\dfrac{e^{-i\omega t}}{r^2}\,\hat n\times\hat p, & r\ll\lambda \end{cases}$$

$$\vec E = \frac{iZ_0}{k}\,\vec\nabla\times\vec H$$

$$E \simeq \frac{iZ_0kc}{4\pi}\,\vec\nabla\times\left|\frac{e^{-i\omega t+ik\cdot r}}{r^2}\right|(r\times p) =$$

$$= \frac{iZ_0kc}{4\pi}\,\frac{e^{-i\omega t+ikr}}{r}\,ik\left(1-\frac{1}{ikr}\right)\hat n\times\hat n\times\hat p =$$

(7.16a) $$= i^2Z_0\,\frac{k^2c}{4\pi}\,(n\times n\times p)\,\frac{e^{-i\omega t+ikr}}{r} = \boxed{Z_0\,\vec H\times\vec n}, \quad r\gg\lambda$$

$$E \simeq \frac{i\omega}{4\pi}\,e^{-i\omega t}\left(\frac{iZ_0}{k}\right)\nabla\times\left(\frac{r\times p}{r^3}\right) = -\frac{c}{4\pi}Z_0\,e^{-i\omega t}\,\nabla\times\left(\frac{r\times p}{r^3}\right) =$$

(7.16b) $$= -\frac{1}{4\pi}\frac{1}{\sqrt{\epsilon_0\mu_0}}\sqrt{\frac{\mu_0}{\epsilon_0}}\, e^{-i\omega t}\, \bar{j}\left(\frac{\bar{r}\times\bar{p}}{r^3}\right) \cdot \boxed{\frac{1}{4\pi\epsilon_0}\frac{3n(p\cdot n)-p}{r^3}}\quad r\ll\lambda$$

for $r \gg \lambda$:

$$S = \frac{1}{2}\mathrm{Re}\left(E\times H^*\right] = \frac{1}{2}\mathrm{Re}\left[Z_0\, H\times n\times H^*\right] = \frac{n\cdot|H|^2 Z_0}{2}$$

$$= \left(\frac{k^2 c}{4\pi}\right)^2 \frac{|p|^2}{r^2}$$

$$\frac{dS}{d\Omega} = \frac{1}{2}\mathrm{Re}\left|r^2 E\times H^*\right] = \frac{Z_0 k^4 c^2}{32\pi^2}\left|(n\times p)\times n\right|^2 =$$

(7.17) $$= \frac{Z_0 k^4 c^2}{32\pi^2}|p|^2 \sin^2\theta$$

$$S_{tot} = \frac{k^4 c^2 Z_0}{32\pi^2}|p|^2\, 2\pi \int \sin^3\theta\, d\theta = \frac{k^4 c^2 Z_0}{12\pi}|p|^2$$

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Next leading terms

$$A = \frac{\mu_0}{4\pi}\frac{e^{-i\omega t + ikr}}{r}\int\left(\frac{\bar{r}\cdot\bar{r}'}{r^2} - i\bar{k}\cdot r'\right) j(r')\, d^3r' =$$

$$= \frac{\mu_0}{4\pi}\frac{e^{-i\omega t+ikr}}{r}\left(\frac{1}{r} - ik\right)\int (n\cdot r')\, j(r')\, d^3r'$$

$$(n\cdot\vec{r}')\,\vec{j}(r') = \frac{1}{2}\left[(n\cdot r')\vec{j} + (n\cdot j)\vec{r}'\right] + \frac{1}{2}(x'\times\vec{j})\times\vec{n}$$

Introducing magnetic moment:

$$\bar{m} = \frac{1}{2}\int \bar{x}'\times\bar{j}\, dx',\ \text{obtain}$$

$$\vec{A}_m = \frac{ik\mu_0}{4\pi}\frac{e^{-i\omega t+ikr}}{r}\left(1 - \frac{1}{ikr}\right)\bar{n}\times\bar{m} \leftarrow \text{magnetic dipole potential}$$

radiation pattern, etc. identical to A_d, ...

except for polarization

Remaining terms:

$$\frac{1}{2}\int\left[(n\cdot r')\vec{j}+(n\cdot j)\vec{r}'\right]d^3r' = -\frac{i\omega}{2}\int r'(\bar{n}\cdot\bar{r}')\rho(x')dx',$$

leading to,

$$\vec{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\,\bar{n}\times\vec{Q}(\vec{n}),$$

$$Q_\alpha(\bar{n}) = \sum_\beta Q_{\alpha\beta}\,n_\beta,\quad Q_{\alpha\beta} = \int(3r_\alpha r_\beta - r^2\delta_{\alpha\beta})\rho(r)\,dr$$

Quadrupole moment