Note Title 2/15/2016

# wave propagation in waveguides

Consider a structure that is homogeneous along the z-direction and is surrounded by a PEC boundary


The field in this structure can be represented as a linear combination of waveguide modes

Homogeneity along the z-axis ⇒ conservation of z-component of the wave vector ⇒

$$E, H \propto e^{i k_z z - i\omega t}$$

In general a mode can be a combination of TE ($E_z = 0$) & TM ($H_z = 0$) components

Consider such solution in Maxwell eq:

$$\begin{cases} \nabla \times \vec{H} = -i\omega\epsilon\vec{E} \\ \nabla \times \vec{E} = i\omega\mu\vec{H} \end{cases}$$

(6.1a)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = -i\omega\epsilon\vec{E} \Rightarrow \begin{cases} \frac{\partial H_z}{\partial y} - ik_z H_y = -i\omega\epsilon E_x \\ \frac{\partial H_z}{\partial x} - ik_z H_x = i\omega\epsilon E_y \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\omega\epsilon E_z \end{cases}$$

(6.1b)

$$\begin{cases} \frac{\partial E_z}{\partial y} - ik_z E_y = i\omega\mu H_x \Rightarrow H_x = \frac{1}{i\omega\mu}\left(\frac{\partial E_z}{\partial y} - ik_z E_y\right) \\ \frac{\partial E_z}{\partial x} - ik_z E_x = -i\omega\mu H_y \Rightarrow H_y = \frac{i}{\omega\mu}\left(\frac{\partial E_z}{\partial x} - ik_z E_x\right) \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\mu H_z \end{cases}$$

$$\frac{\partial H_z}{\partial y} - ik_z \frac{i}{\omega\mu}\left(\frac{\partial E_z}{\partial x} - ik_z E_x\right) = -i\omega\epsilon E_x$$

$$\frac{\partial H_z}{\partial x} - ik_z \frac{1}{i\omega\mu}\left(\frac{\partial E_z}{\partial y} - ik_z E_y\right) = i\omega\epsilon E_y$$

(6.2a)

$$\boxed{\begin{aligned} E_x &= \frac{i\omega\mu}{\omega^2\epsilon\mu - k_z^2}\left(\frac{\partial H_z}{\partial y} + \frac{k_z}{\omega\mu}\frac{\partial E_z}{\partial x}\right) \\ E_y &= \frac{i\omega\mu}{\omega^2\epsilon\mu - k_z^2}\left(-\frac{\partial H_z}{\partial x} + \frac{k_z}{\omega\mu}\frac{\partial E_z}{\partial y}\right) \end{aligned}}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \frac{\cancel{-i\omega}\mu}{\omega^2\epsilon\mu - k_z^2}\left(\frac{\partial^2 H_z}{\partial x^2} \cancel{- \frac{k_z}{\omega\mu}\frac{\partial^2 E_z}{\partial y\,\partial x}} + \frac{\partial^2 H_z}{\partial y^2} \cancel{+ \frac{k_z}{\omega\mu}\frac{\partial^2 E_z}{\partial x\,\partial y}}\right) = i\cancel{\omega}\mu H_z$$

(6.3a)

$$\Rightarrow \boxed{\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + (\omega^2\epsilon\mu - k_z^2)H_z = 0}$$

Similarly

$$E_x = \frac{i}{\omega\epsilon}\left(\frac{\partial H_z}{\partial y} - ik_z H_y\right) \quad \Big| \quad E_y = \frac{-i}{\omega\epsilon}\left(\frac{\partial H_z}{\partial x} - ik_z H_x\right)$$

$$\frac{\partial E_z}{\partial y} - \cancel{i}k_z \frac{\cancel{(-i)}}{\omega\epsilon}\left(\frac{\partial H_z}{\partial x} - ik_z H_x\right) = i\omega\mu H_x \qquad \left(i\omega\mu - i\frac{k_z^2}{\omega\epsilon}\right)$$

$$\frac{\partial E_z}{\partial x} - ik_z \frac{i}{\omega\epsilon}\left(\frac{\partial H_z}{\partial y} - ik_z H_y\right) = -i\omega\mu H_y \qquad \left(-i\omega\mu + i\frac{k_z^2}{\omega\epsilon}\right)$$

(6.2b)

$$\Rightarrow \boxed{\begin{aligned} H_x &= \frac{i\omega\epsilon}{\omega^2\epsilon\mu - k_z^2}\left(\frac{k_z}{\omega\epsilon}\frac{\partial H_z}{\partial x} - \frac{\partial E_z}{\partial y}\right) \\ H_y &= \frac{i\omega\epsilon}{\omega^2\epsilon\mu - k_z^2}\left(\frac{k_z}{\omega\epsilon}\frac{\partial H_z}{\partial y} + \frac{\partial E_z}{\partial x}\right) \end{aligned}}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{i\omega\epsilon}{\omega^2\epsilon\mu - k_z^2}\left[\frac{k_z}{\omega\epsilon}\frac{\partial H_z}{\partial x \partial y} - \frac{k_z}{\omega\epsilon}\frac{\partial H_z}{\partial y \partial x} + \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2}\right]$$

$$(6.3b)\quad \boxed{\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (\omega^2\epsilon\mu - k_z^2)E_z = 0}$$

$\Rightarrow$ Both $E_z$ & $H_z$ satisfy Helmholtz eqns,

In-plane components of $\vec{E}$ ($\vec{H}$) are proportional to 2D grad $E_z$ ($H_z$) & 2D curl of $H_z$ ($E_z$)

Some guides support propagation of TE ($E_z = 0$, $H_z \neq 0$) & TM ($E_z \neq 0$, $H_z = 0$) modes,

Some support propagation of TEM ($E_z = H_z = 0$) modes

Optical guides support HE (TM-like) & EH (TE-like) modes that have both $E_z, H_z \neq 0$

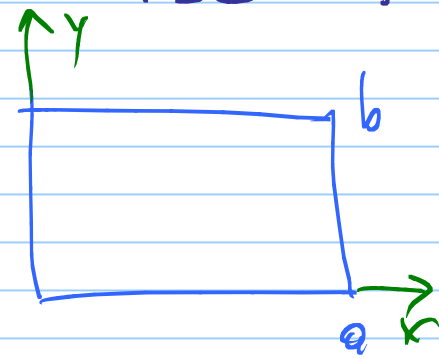
Note: we solve for $E_z, H_z$-field distributions, once these are known, all other field components are calculated.

Contrast with plane-wave retraction:

| | plane wave | waveguide |
|---|---|---|
| prop: | $x$-$z$ | $z$ |
| TE | $E \parallel y$ | $E_z = 0$ |
| TM | $H \parallel y$ | $H_z = 0$ |

We now look @ propagation in some representative waveguides:

① Rectangular guides with PEC walls:




(63) $\Rightarrow \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = -(\omega^2 \epsilon\mu - k_z^2) f$

where $f = E_z, H_z$

Assume that $f = X(x) \cdot Y(y)$

$$\frac{X''}{X} + \frac{Y''}{Y} = -\omega^2\epsilon\mu + k_z^2 ;$$

(6.4) $\dfrac{X''}{X} = -q_x^2$ ; $\dfrac{Y''}{Y} = -q_y^2 \Rightarrow k_z^2 = \omega^2\epsilon\mu - q_x^2 - q_y^2$

TM modes: $E_z\Big|_{x=a} = E_z\Big|_{y=b} = E_z\Big|_{x=0} = E_z\Big|_{y=0} = 0 \Rightarrow$

$E_z = \sin q_x^n x \, \sin q_y^m y,$

(6.5) $q_x^n = \dfrac{\pi}{a} n$ ; $q_y^m = \dfrac{\pi}{b} m$, $n, m \in \mathbb{N}$ ($n, m \neq 0!$)

$$k_z^{(n,m)^2} = \omega^2 \epsilon\mu - \left(\frac{\pi}{a} n\right)^2 - \left(\frac{\pi}{b} m\right)^2$$

$$k_z = \sqrt{\frac{\omega^2}{c^2}\epsilon_r\mu_r - \left(\frac{\pi}{a} n\right)^2 - \left(\frac{\pi}{b} m\right)^2}$$

phase velocity $v_p = \dfrac{\omega}{k_z} = \dfrac{\omega}{\sqrt{\frac{\omega^2}{c^2}\epsilon_r\mu_r - \left(\frac{\pi}{a} n\right)^2 - \left(\frac{\pi}{b} m\right)^2}} =$

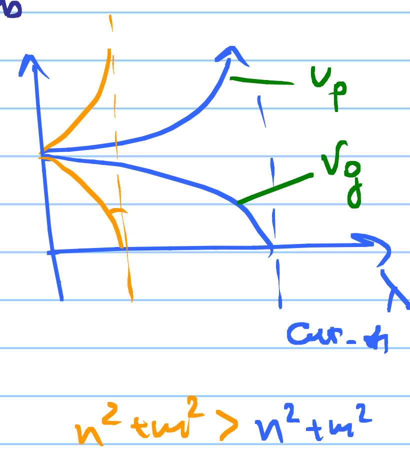
$$= \frac{c}{\sqrt{\epsilon_r\mu_r - \left(\frac{\pi\lambda}{a\,2\pi} n\right)^2 - \left(\frac{\lambda}{2b} m\right)^2}}$$

(6.6a) $v_p = \frac{c}{\sqrt{\epsilon_r\mu_r - \left(\frac{\lambda}{2a}n\right)^2 - \left(\frac{\lambda}{2b}m\right)^2}}$ , $\lambda = \frac{2\pi c}{\omega}$ – vacuum wavel.

group velocity: $v_g = \frac{d\omega}{dk} = 1\Big/\frac{dk}{d\omega} = 1\Big/\frac{2\epsilon_r\mu_r\frac{\omega}{c^2}}{2\sqrt{\ldots}} = \frac{k_z c}{\frac{\omega}{c}\epsilon_r\mu_r}$

(6.6b) $v_g = \frac{c}{\epsilon_r\mu_r}\sqrt{\epsilon_r\mu_r - \left(\frac{\lambda}{2a}n\right)^2 - \left(\frac{\lambda}{2b}m\right)^2}$

(6.6c) Note: $\boxed{v_p v_g = \frac{c^2}{\epsilon_r\mu_r}}$




(assuming that $\epsilon_r, \mu_r \neq f(\omega)$)

plot fields, show that B.C. satisfied!

TM-waves:

$$E_y\big|_{x=0} = E_y\big|_{x=a} = E_x\big|_{y=0} = E_x\big|_{y=b} = 0$$

Note that $E_x \propto \frac{\partial H_z}{\partial y}$ ; $E_y \propto \frac{\partial H_z}{\partial x}$

(6.7) $\Rightarrow$ $\boxed{H_z = \cos q_x^n x \cos q_y^m y \;;\; q_x^n = \frac{\pi n}{a},\; q_y^m = \frac{\pi m}{b} \quad n, m = 0, 1, \ldots \;;\; \text{but } n^2+m^2 > 0}$

(b) planar guide, PEC walls: ($b \to \infty$)

TE modes

$$E_z = \sin(q_x^n x)\ ; \quad q_x^n = \frac{\pi n}{a}, \quad n = 1, \dots$$

(6.8) $$k_z^2 = \frac{\omega^2}{c^2}\epsilon_r\mu_r - \left(\frac{\pi n}{a}\right)^2$$

TM modes:

(6.8b) $$H_z = \cos(q_x^n x)\ ; \quad q_x^n = \frac{\pi n}{a}, \quad n = 0, 1, \dots$$

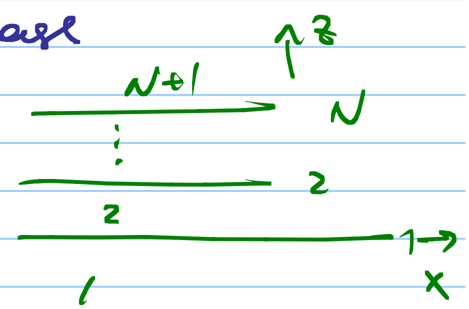
(6.9c) TEM mode $H_z = 1$, $E_y = 0$, $E_x = \text{const}$

$\exists$ in non-simply connected geometries

(c) planar guide, general case

(6.3) in each layer $\Rightarrow$

(Note change in geometry)




(6.10) $$\Delta_2 f = \frac{\partial^2 f}{\partial z^2} = \left(-\omega^2 \epsilon\mu + k_x^2\right) f$$

$$\Rightarrow f = e^{\pm i k_x x}$$

$\Rightarrow$ map onto TMM:

$$k_x^2 > \epsilon_1\mu_1\omega^2, \ \epsilon_{N+1}\mu_{N+1}\omega^2$$

"incident" beam = 0 ; only transmitted and reflected beams exist $\Rightarrow$

$$\begin{pmatrix} t \\ 0 \end{pmatrix} = T_{tot}(k_x, \omega) \begin{pmatrix} 0 \\ r \end{pmatrix}$$

$\Rightarrow$ disp. relation: $\left\{ \begin{array}{l} \left[T_{tot}(k_x,\omega)\right]_{2,2} = 0 \\ \det \left| T_{tot}(k_x,\omega)_{22} \right| = 0 \end{array} \right.$

(6.11) (for 4x4 matrices):

Once dispersion is calculated, mode profile is given by "eigenvectors"

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Once dispersion and profiles of eigenmodes are calculated, the field profile of the signal propagating inside the waveguide is given by

$$(E, H) = \sum_{n,m} \alpha_{nm} E, H^{n,m}(x,y)\, e^{-i\omega t + i k_z^{(n,m)} z}$$

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Consider, for example

$$E_x(z=0) = 1 \quad \text{in TE waves}$$

$$H_z = \cos\left(\frac{\pi n}{L_x} x\right) \cos\left(\frac{\pi m}{L_y} y\right)$$

$$E_x \propto \frac{\partial H_z}{\partial y} = -\frac{\pi m}{L_y} \cancelto{1}{\cos\left(\frac{\pi n}{L_x} x\right)} \sin \frac{\pi m}{L_y} y$$

$$E_y \propto \frac{\partial H_z}{\partial x} = \cancelto{n=0}{-\frac{\pi n}{L_x} \sin\left(\frac{\pi n}{L_x} x\right)} \cos\left(\frac{\pi m}{L_y} y\right)$$

$$\int E_x = \sum \frac{a_m}{q_m^2} q_m \sin \frac{\pi m}{L_y} y = 1 \qquad \sin \frac{\pi l}{L_y} y\, dy$$

$$\frac{a_l}{q_l} \frac{1}{2L_y} = \frac{L_y}{\pi l} \left(\cos \pi l - 1\right)$$

$$\frac{a_\ell}{2\cancel{\pi\ell}} = \frac{L_y}{\cancel{\pi\ell}}\left((-1)^\ell - 1\right)$$

$$\boxed{a_\ell = 2L_y\left((-1)^\ell - 1\right)}$$