Note Title 1/26/2016

Any finite wave packet can be represented as a linear combination of plane waves.

Consider first one-dimensional propagation

$$\vec{E}(x,z,t) = \int E(\omega)\, e^{-i\omega t + i k_z z}\, d\omega$$

@ $t=0$: $E(z,0) = \int E(k)\, e^{ikz}\, dk$

@ $z=0$: $E(0,t) = \int E(\omega)\, e^{-i\omega t}\, d\omega$

$$\Rightarrow E(k) = \frac{1}{2\pi}\int E(z,0)\, e^{-ikz}\, dz; \quad E(\omega) = \frac{1}{2\pi}\int E(0,t)\, e^{i\omega t}\, dt$$

with $k = n\omega$

Assume that

$$k_z = n(\omega)\frac{\omega}{c}$$

$$E(z,t) = \int E(k)\, e^{ikz - i\omega t}\, dk$$

$$\omega = \omega_0 + \frac{d\omega}{dk}(k-k_0)$$

$$E(z,t) = \int E(k)\, e^{-i\omega_0 t + ikz - i\frac{d\omega}{dk}kt + i\frac{d\omega}{dk}k_0 t}\, dk =$$

$$= \underbrace{e^{-i\omega_0 t + i\left.\frac{d\omega}{dk}\right|_{k_0} k_0 t}}_{\text{phase factor}} \underbrace{\int E(k)\, e^{ik\left(z - \left.\frac{d\omega}{dk}\right|_{k_0} t\right)}\, dk}_{E(z', t=0)}$$

$$z' = z - \left.\frac{d\omega}{dk}\right|_{k_0} t \;; \qquad \frac{d\omega}{dk} = v_g \text{ – group velocity}$$

in 3D $\quad \vec{v}_g = \frac{d\omega}{d\vec{k}} = \left\{ \frac{\partial \omega}{\partial k_x}\hat{x}, \frac{\partial \omega}{\partial k_y}\hat{y}, \frac{\partial \omega}{\partial k_z}\hat{z} \right\}$

Note:

→ in highly-dispersive materials

v_g could be $>c$, or even <0 (pulse reshaping)

→ v_g does not represent velocity of information flow. Information flows @ c

→ Gaussian pulse: v_g - velocity of max

Numerical Implementation: DFT

$$E(\omega_j) = \frac{1}{2\pi}\int E(t)\, e^{i\omega t} dt = \frac{1}{2\pi}\sum_m E(t_m) e^{i\omega_j t_m}\Delta t =$$

$$= \frac{T}{2\pi N}\sum_m E(t_m) e^{i\omega_j Tm/N} = F_{jm}\, E(t_m)$$

$$t_m = \frac{T}{N}m; \qquad F_{jm} = \frac{T}{2\pi N} e^{i j \Omega T m/N^2}$$

$$\omega_j = \frac{\Omega}{N} j; \; \Omega = \frac{2\pi N}{T}; \; \tilde{F}_{nj} = \frac{\Omega}{N} e^{-ijn\frac{\Omega T}{N^2}}$$

$$E(t_n) = \int E(\omega_j) e^{-i\omega_j t_n} d\omega_j = \frac{\Omega}{N}\sum e^{-ijn\frac{\Omega T}{N^2}} E(\omega_j)$$

$$\tilde{F}_{nj} F = \frac{T\Omega}{2\pi N^2}\sum_j e^{ij\frac{\Omega T}{N^2}(m-n)} = \frac{1}{N}\sum_{j=0}^{N-1} e^{ij\frac{2\pi(m-n)}{N}} =$$

$$= \frac{1}{N}\frac{1 - e^{i\frac{N}{N}2\pi(m-n)}}{1 - e^{i\frac{2\pi(m-n)}{N}}} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

Show:

→ pulse propagation through dispersive medium ($v_g < 0$?)

→ Goos-Hänchen shift

→ Talk about periodicity due to DFT