Analytical properties of permittivity

Note Title 1/25/2016

In reality all fields inside materials are real. Here, we focus on permittivity.

All calculations can be repeated for permeability

In a general material,

(3.1) $$D(t) = \varepsilon_0 E(t) + P(t).$$

In general, the relationship must be causal, i.e. only preceding fields can affect current fields. Therefore,

(3.2) $$D(t) = \varepsilon_0 \left[E(t) + \int_0^\infty G(\tau)\, E(t-\tau)\, d\tau\right]$$

where $G(\tau)$ is some finite function that

$$G(\tau) \to 0 \quad \text{for} \quad \tau \to \infty$$

The time-domain dynamics considered above can be converted to the frequency domain via Fourier transform.

Basic Fourier transform:

(3.3) $$\begin{cases} F(\omega) = \displaystyle\int_{-\infty}^{\infty} f(t) e^{+i\omega t} dt \\ f(t) = \displaystyle\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, dt \end{cases} \qquad \frac{1}{2\pi}$$

Multiplying (3.2) by $\frac{1}{2\pi} e^{i\omega t}$, and integrating, obtain:

$$\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} D(t)\, e^{i\omega t} dt}_{D(\omega)} = \frac{\epsilon_0}{2\pi}\left[\underbrace{\int_{-\infty}^{\infty} E(t) e^{+i\omega t} dt}_{E(\omega)} + \right.$$

$$\left. + \underbrace{\int_{-\infty}^{\infty} dt \int_0^{\infty} d\tau\, G(\tau)\, E(t-\tau) e^{i\omega t}}\right]$$

$$\to \int_0^{\infty} d\tau\, f(\tau) \underbrace{\int_{-\infty}^{\infty} d(t-\tau)\, E(t-\tau)\, e^{i\omega(t-\tau)}}_{2\pi E(\omega)} e^{i\omega\tau}$$

(3.4)
$$\begin{cases} D(\omega) = \epsilon_0 E(\omega)\left[1 + \int_0^{\infty} G(\tau) e^{i\omega\tau} d\tau\right] = \\ = \epsilon_0\, \epsilon(\omega)\, E(\omega), \text{ where} \\ \epsilon(\omega) = 1 + \int_0^{\infty} G(\tau) e^{i\omega\tau} d\tau;\ G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\epsilon(\omega) - 1]\, e^{-i\omega t} dt \end{cases}$$

Note that F.T essentially provides the expansion of an arbitrary pulse in terms of plane waves:

(3.5)
$$\{D, \ldots\}(t) = \int_{-\infty}^{\infty} \{D(\omega), \ldots\} e^{-i\omega t} d\omega$$

Note time convention!

Let's now focus on the properties of $\epsilon(\omega)$

1. $f(\tau)$ is finite for $\tau > 0$
2. $e^{i\omega t}$ is finite for real ω, $e^{i\omega t} \to 0$ when $im(\omega) > 0$

$\Rightarrow$ $\epsilon(\omega)$ does not have discontinuities for $im(\omega) > 0$

Note: $im(\omega) > 0$ represents growing fields, see (3.5)

From (3.4); ($f, \tau \in \mathbb{R}$)

$$\epsilon(-\omega) = 1 + \int_0^{\infty} f(\tau) e^{-i\omega\tau} d\tau = [\epsilon(\omega^*)]^*$$

for $\omega \in \mathbb{R}$: $\quad \epsilon(-\omega) = \epsilon(\omega)^*$

$$\Rightarrow \epsilon'(-\omega) + i\epsilon''(-\omega) = \epsilon'(\omega) - i\epsilon''(\omega)$$

$$\Rightarrow \epsilon(\omega) \text{ even} = c + \#\omega^2 + \#\omega^4, \ldots$$

Note that $\epsilon''(\omega > 0) > 0$

Consider now a limit of small frequencies

$$\nabla \times H = \epsilon_0 \epsilon(\omega) \frac{\partial \vec{E}}{\partial t} = -i\omega \epsilon_0 \epsilon \vec{E} \quad \text{(in generic medium)}$$

$$\nabla \times H = j = \sigma \vec{E} \quad \text{(in D.C. conductors)}$$

in the limit $\omega \to 0$: $-i\omega\epsilon_0\epsilon = \sigma$

(3.6) $$\Rightarrow \epsilon(\omega) \to \frac{\sigma i}{\epsilon_0 \omega}$$

$$\Rightarrow \epsilon''(0) = \frac{\sigma i}{\epsilon_0 \omega} + \#\omega + \#\omega^3 + \ldots$$

Since $\epsilon(\omega)$ is analytical in the upper half-plane,

pole on axis

$$\epsilon(\omega) = \epsilon_0 + \frac{1}{\pi i} \oint \frac{\epsilon(\omega) - \epsilon_0}{\omega - \omega_0} d\omega = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\epsilon(\omega)\, d\omega}{\omega - \omega_0}$$

(3.7)
$$\begin{cases} \epsilon'(\omega_0) = \epsilon_0 + \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\epsilon''(\omega)\, d\omega}{\omega - \omega_0} \\ \epsilon''(\omega_0) = -\dfrac{1}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\epsilon'(\omega) - \epsilon_0}{\omega - \omega_0} d\omega \end{cases}$$

$$\epsilon'(\omega_0) = \epsilon_0 + \frac{1}{\pi}\left[\int_{-\infty}^{0} \frac{\epsilon''(\omega)\,d\omega}{\omega-\omega_0} + \int_{0}^{\infty} \frac{\epsilon''(\omega)\,d\omega}{\omega-\omega_0}\right] =$$

$$= \epsilon_0 + \frac{1}{\pi}\int_0^{\infty} \epsilon''(\omega)\,d\omega \left[\frac{1}{(\omega+\omega_0)} + \frac{1}{\omega-\omega_0}\right] =$$

$$= \epsilon_0 + \frac{1}{\pi}\int_0^{\infty} \frac{\epsilon''(\omega)\,d\omega}{\omega^2-\omega_0^2}\left[\omega-\omega_0+\omega+\omega_0\right] =$$

(3.7b) $$\epsilon'(\omega_0) = \epsilon_0 + \frac{2}{\pi}\int_0^{\infty} \frac{\omega\epsilon''(\omega)\,d\omega}{\omega^2-\omega_0^2}$$

Note that by measuring absorption spectrum we can recover complete $\epsilon(\omega)$!

Similarly,

$$\epsilon''(\omega_0) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\epsilon'(\omega)-\epsilon_0}{\omega-\omega_0}\,d\omega = -\frac{1}{\pi}\int_0^{\infty}(\epsilon'(\omega)-\epsilon_0)\,d\omega \times$$

$$\times\left(\frac{1}{\omega-\omega_0} - \frac{1}{\omega-\omega_0}\right) + \frac{\sigma}{\epsilon_0\omega} \Rightarrow$$

(3.7c) $$\epsilon''(\omega_0) = \frac{\sigma}{\epsilon_0\omega} - \frac{2\omega}{\pi}\int_0^{\infty} \frac{\epsilon'(\omega)-\epsilon_0}{\omega^2-\omega_0^2}\,d\omega$$

Consider now a model of oscillating electron inside the material. Its equation of motion

(3.8) $$m\ddot{x} = eE_0e^{-i\omega t} - m\gamma_0\dot{x} - m\omega_0^2 x;\quad \text{Assuming } x = x_0e^{-i\omega t}$$

$$(-m\omega^2 - i\gamma_0\omega + m\omega_0^2)\,x_0 = eE_0$$

(3.9) $$x_0 = \frac{eE_0}{m}\,\frac{1}{\omega_0^2-\omega^2-i\gamma_0\omega} = \frac{eE_0}{m}\,\frac{\omega_0^2-\omega^2+i\gamma_0\omega}{(\omega_0^2-\omega^2)^2+\gamma_0^2}$$

(3.10) $$P = ex_0N = \epsilon_0\chi E$$

(3.11) $\Rightarrow \quad \epsilon = \epsilon_0 + \frac{e^2 N}{m} \frac{\omega_0^2 - \omega^2 + i\gamma_0\omega}{(\omega_0^2-\omega^2)^2 + \gamma_0^2}$

for free electrons: $(\omega_0 = 0)$

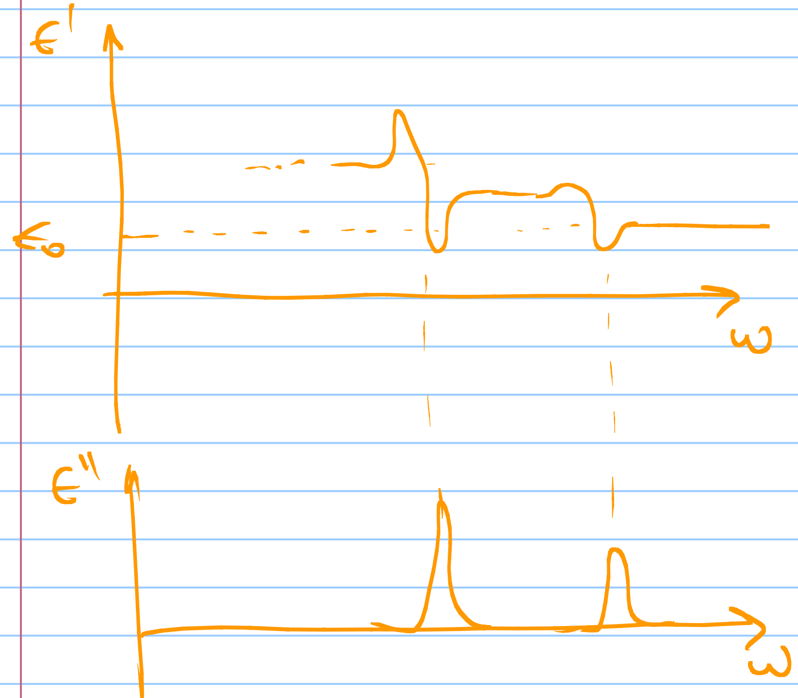
$$\epsilon = \epsilon_0 - \frac{e^2 N}{m}\frac{1}{\omega^2 + i\gamma_0\omega} = \epsilon_0 - \frac{e^2 N}{m\omega}\frac{\omega - i\gamma_0}{\omega^2+\gamma_0^2} =$$

(3.11b) $$= \epsilon_0 - \frac{e^2 N}{m(\omega^2+\gamma_0^2)} + i\frac{e^2 N\gamma_0}{m\,\omega(\omega^2+\gamma_0^2)}$$

for multiple resonant frequencies:

(3.11c) $\epsilon = \epsilon_0 + \frac{e^2 N}{m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$; f_j - resonator strength

(3.12) Sum rule : $\sum f_j = 1$ $\left(N\sum_j f_j = \#e^- \text{ in unit volume}\right)$



In continuous limit $\frac{e^2 N}{m} f(\omega) \sim \frac{2\epsilon''(\omega)}{\pi}\,\omega$