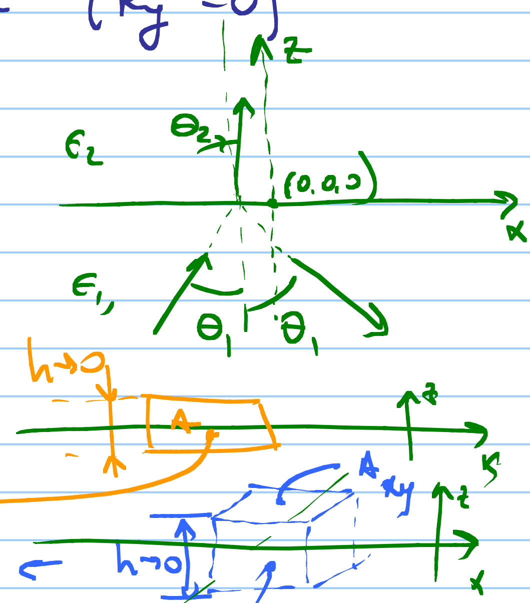
EM II lecture 02 - reflection and transmission

In the previous classes we have shown that plane wave represents a solution of Maxwell equations in a homogeneous material. We now consider an interface between two materials. We will assume that the interface is in xy plane and that the light propagates in xz plane ($k_y = 0$)

We again start with Maxwell equations ($\mu = 1$)



(2.1a) $$\begin{cases} \int \vec{\nabla} \times \vec{H} = \epsilon_0 \hat{\epsilon} \dfrac{\partial \vec{E}}{\partial t} \\ \int \vec{\nabla} \times \vec{E} = -\cancel{\mu_0} \dfrac{\partial \vec{H}}{\partial t} \end{cases} dA$$

(2.1b) $$\begin{cases} \int \operatorname{div} \vec{B} = 0 \\ \int \operatorname{div} \vec{D} = 0 \end{cases} d^3r$$

$$\int \vec{\nabla} \times \vec{H}\, dl = \oint H\, dl = \left(H_{2_\tau} - H_{1_\tau}\right) l = \epsilon_0 \epsilon \int \frac{\partial E}{\partial t} dA \to 0$$

(2.2a) $$\begin{cases} \Rightarrow H_{2_\tau} = H_{1_\tau} \\ \Rightarrow E_{2_\tau} = E_{1_\tau} \end{cases}$$

$$\int \operatorname{div} B\, d^3r = \oint \vec{B} \cdot d^2\vec{a} = \left(B_{2_n} - B_{1_n}\right) A_{xy} = 0$$

(2.2b) $$\begin{cases} \Rightarrow B_{2n} = B_{1n} \\ \Rightarrow D_{2n} = D_{1n} \end{cases}$$

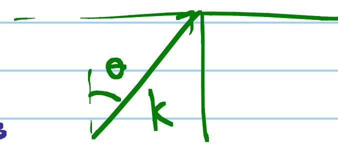
Consider now a situation when a plane wave incident onto the interface, and excites one refracted and one reflected plane wave

For a plane wave,

(2.3) $$\vec{E}, \vec{B} = (\vec{E}_0, \vec{B}_0)\cdot e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

Assume that boundary conditions are satisfied @ $\vec{r} = \vec{0}$. then the b.c. will be satisfied if k_x of all the waves is the same.

Assuming that "i", "r", "t" represent incident, reflected, transmitted waves,



(2.4)
$$\left\{\begin{array}{l} k_{i_x} = k_{r_x} = k_{t_x} \\ \Rightarrow k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t \\ \Rightarrow \sin\theta_i = \sin\theta_r \quad \Rightarrow \theta_i = \theta_r \\ \Rightarrow k_1 \sin\theta_i = k_2 \sin\theta_t \overset{(\text{isotr.})}{\Longrightarrow} \cancel{\frac{\omega}{c}}\, n_1 \sin\theta_i = \cancel{\frac{\omega}{c}}\, n_2 \sin\theta_t \end{array}\right.$$

Note, that conservation of k_x $\Leftarrow$ homogeneous interface

(Snell's law)

We now use boundary conditions to calculate transmission & reflection amplitudes.

In isotropic & uniaxial media,

TE modes (s): $\bar{E} \parallel \hat{y}$; $\vec{H} \perp \hat{y}$

TM mode (p) $\bar{E} \perp \hat{y}$; $\vec{H} \parallel \hat{y}$

$\Rightarrow$ polarizations don't mix

In general, however they do. We will therefore consider general case:

(2.5)
$$E_i = a_{i1}^+ \vec{E}_{\parallel}^+ e^{i\vec{k}_{\parallel}\cdot\vec{r} - i\omega t} + a_{i2}^+ \vec{E}_{12}^+ e^{i\vec{k}_{12}\cdot\vec{r} - i\omega t}$$

And similar for E_t, E_r, B_i, B_r, B_t

For uniaxial medium:

$$\vec{E}_{\parallel}^{\pm} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; \quad \vec{E}_{12}^{\pm} = \begin{bmatrix} \pm k_{z12}\epsilon_{z,1} \\ 0 \\ -k_x \epsilon_{\perp,1} \end{bmatrix} ;$$

$$\vec{H}_{\parallel}^{\pm} = \frac{1}{\mu\omega}\begin{bmatrix} \mp k_z \\ 0 \\ k_x \end{bmatrix} ; \quad \vec{H}_{12}^{\pm} = \frac{1}{\mu\omega}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_x & 0 & \pm k_z \\ \pm k_z \epsilon_{z,1} & 0 & -k_x\epsilon_{\perp,1} \end{vmatrix} =$$

$$= \frac{1}{\mu\omega}\begin{bmatrix} 0 \\ k_x^2\epsilon_1 + k_z^2\epsilon_2 \\ 0 \end{bmatrix} = \frac{\omega\epsilon_1\epsilon_2}{c^2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Overall the 4 boundary conditions (continuity of E_t, H_t) can be written as:

E_x $$k_{z12}\epsilon_{2,1}(a_{i2} - a_{r2}) = k_{z22}\epsilon_{2,2}a_{t2}$$

E_y $$a_{i1} + a_{r1} = a_{t1}$$

H_x $$-\frac{k_{z11}}{\mu_1\omega}(a_{i1} - a_{r1}) = \frac{k_{z21}}{\mu_2\omega}a_{t1}$$

H_y $$\frac{\omega\epsilon_{1,1}\epsilon_{2,1}}{c^2}(a_{i2} + a_{r,2}) = \frac{\omega}{c^2}\epsilon_{1,2}\epsilon_{2,2}(a_{t,2})$$

(2.6)
$$\begin{bmatrix} 0 \\ 1 \\ -\frac{k_{z1}^{(i)}}{\mu^{(1)}\omega} \\ 0 \end{bmatrix} a_1^{(i)} + \begin{bmatrix} k_{z2}^{(1)}\epsilon_2^{(1)} \\ 0 \\ 0 \\ \frac{\omega}{c^2}\epsilon_1^{(1)}\epsilon_2^{(1)} \end{bmatrix} a_2^{(i)} + \begin{bmatrix} 0 \\ 1 \\ \frac{k_{z1}^{(1)}}{\mu^{(1)}\omega} \\ 0 \end{bmatrix} a_1^{(r)} + \begin{bmatrix} -k_{z2}^{(1)}\epsilon_2^{(1)} \\ 0 \\ 0 \\ \frac{\omega}{c^2}\epsilon_1^{(1)}\epsilon_2^{(1)} \end{bmatrix} a_2^{(r)} =$$

$$
\begin{bmatrix} 0 \\ 1 \\ -\dfrac{k_{z2}^{(2)}}{\mu^{(2)}\omega} \\ 0 \end{bmatrix} Q_1^{(t)} + \begin{bmatrix} k_{z2}^{(2)}\epsilon_2^{(2)} \\ 0 \\ 0 \\ \dfrac{\omega}{c^2}\epsilon_1^{(2)}\epsilon_2^{(2)} \end{bmatrix} Q_2^{(t)}
$$

Rearrange (2.6)

$$
\underbrace{\begin{bmatrix} 0 & -k_{z2}^{(1)}\epsilon_2^{(1)} & 0 & -k_{z2}^{(2)}\epsilon_2^{(2)} \\ 1 & 0 & 1 & 0 \\ \dfrac{k_{z1}^{(1)}}{\mu^{(1)}\omega} & 0 & \dfrac{k_{z2}^{(2)}}{\mu^{(2)}\omega} & 0 \\ 0 & \dfrac{\omega}{c^2}\epsilon_1^{(1)}\epsilon_2^{(1)} & 0 & \dfrac{\omega}{c^2}\epsilon_1^{(2)}\epsilon_2^{(2)} \end{bmatrix}}_{M_1} \begin{bmatrix} Q_1^r \\ Q_2^r \\ Q_1^t \\ Q_2^t \end{bmatrix} = \tag{2.7}
$$

$$
\underbrace{\begin{bmatrix} 0 & -k_{z2}^{(1)}\epsilon_2^{(1)} \\ 1 & 0 \\ \dfrac{k_{z1}^{(1)}}{\mu^{(1)}\omega} & 0 \\ 0 & \dfrac{\omega}{c^2}\epsilon_1^{(1)}\epsilon_2^{(1)} \end{bmatrix}}_{M_2} \begin{bmatrix} Q_1^i \\ Q_2^i \end{bmatrix}
$$

scat (reflect) matrix

$$
\Rightarrow \begin{bmatrix} S \\ T_r \end{bmatrix} = \left(M_1^{-1} \cdot M_2\right), \qquad \begin{bmatrix} Q_1^r \\ Q_2^r \\ Q_1^t \\ Q_2^t \end{bmatrix} = \left(M_1^{-1} \cdot M_2\right) \begin{bmatrix} Q_1^i \\ Q_2^i \end{bmatrix} \tag{2.8}
$$

transmission matrix

Matrices $\hat{S}$ and $\hat{T_r}$ solve the problem of reflection/transmission of a plane wave from the interface that is @ $z=0$.

Consider now a case when not one, but two plane waves enter the interface (with same kx)

Then,

(2.8b)
$$\begin{bmatrix} a_1^r \\ a_2^r \\ a_1^t \\ a_2^t \end{bmatrix} = \left(M_1^{-1} \cdot \tilde{M}_2\right) \begin{bmatrix} a_1^i \\ a_2^i \\ \tilde{a}_1^i \\ \tilde{a}_2^i \end{bmatrix}$$

with

(2.7b)
$$\tilde{M}_2 = \begin{bmatrix} 0 & -k_{z2}^{(1)} \epsilon_2^{(1)} & 0 & -k_{z2}^{(2)} \epsilon_2^{(2)} \\ 1 & 0 & 1 & 0 \\ \frac{k_{z1}^{(1)}}{\mu^{(1)}\omega} & 0 & \frac{k_{z1}^{(2)}}{\mu^{(2)}\omega} & 0 \\ 0 & \frac{\omega}{c^2} \epsilon_1^{(1)} \epsilon_2^{(1)} & 0 & \frac{\omega}{c^2} \epsilon_1^{(2)} \epsilon_2^{(1)} \end{bmatrix}$$

Consider now the interface that is @ $z = z_0$. then B.C. (2.7) are replaced by

$$M_1 \begin{bmatrix} e^{-ik_{z1}^{(1)} z_0} a_1^r \\ e^{-ik_{z2}^{(1)} z_0} a_2^r \\ e^{ik_{z1}^{(2)} z_0} a_1^t \\ e^{ik_{z2}^{(2)} z_0} a_2^t \end{bmatrix} = \tilde{M}_2 \begin{bmatrix} e^{ik_{z1}^{(1)} z_0} a_1^i \\ e^{ik_{z2}^{(1)} z_0} a_2^i \\ e^{-ik_{z1}^{(2)}} \tilde{a}_1^i \\ e^{-ik_{z2}^{(2)}} \tilde{a}_2^i \end{bmatrix}$$

or, introducing a matrix $F = \begin{bmatrix} e^{ik_{z1}^{(1)} z_0} & 0 & 0 & 0 \\ 0 & e^{ik_{z2}^{(1)} z_0} & 0 & 0 \\ 0 & 0 & e^{-ik_{z1}^{(2)} z_0} & 0 \\ 0 & 0 & 0 & e^{-ik_{z2}^{(2)} z_0} \end{bmatrix}$

we obtain:

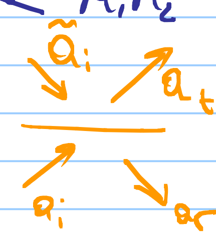
$$M_1 \hat{F}^{-1} \begin{bmatrix} Q_1^r \\ Q_2^r \\ Q_1^t \\ Q_2^t \end{bmatrix} = \tilde{M}_2 \hat{F} \begin{bmatrix} Q_1' \\ Q_2' \\ \tilde{Q}_1' \\ \tilde{Q}_2' \end{bmatrix}$$

or,

(2.8c) $$\begin{bmatrix} Q_1^r \\ Q_2^r \\ Q_1^t \\ Q_2^t \end{bmatrix} = \hat{F} M_1^{-1} \tilde{M}_2 \hat{F} \begin{bmatrix} Q_1' \\ Q_2' \\ \tilde{Q}_1' \\ \hat{Q}_2' \end{bmatrix}$$

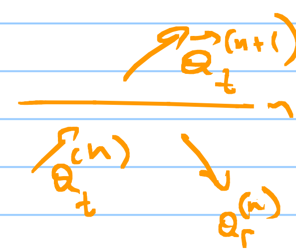
We have developed Scat. matrix formalism that can be used to calculate T/R from multilayer stacks:

(1) $\forall$ interface calculate the matrix $M_1 \tilde{M}_2$ and split it as:



(2.9a) $$M_1 \tilde{M}_2 = \begin{bmatrix} \hat{R}_b & \hat{T}_t \\ \hat{T}_b & \hat{R}_t \end{bmatrix}$$

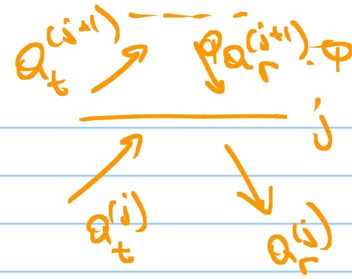
(2) Consider "last" interface



(2.9b) $$\hat{T}^{(n)} = \hat{T}_b^{(n)} \Leftrightarrow \vec{Q}_t^{(n+1)} = T_b^{(n)} Q_i^{(n)}$$

$$\hat{R}^{(n)} = \hat{R}_b^{(n)} \qquad Q_r^{(n)} = R_b^{(n)} Q_t^{(n)}$$

(3) Iterate down:



$$\tilde{Q}_i = \hat{\varphi} R_b^{(j+1)} \hat{\varphi} Q_t^{(j+1)}$$

(2.9)

$$\begin{cases} Q_t^{(j+1)} = T_b^{(j)} Q_t^{(j)} + R_t^{(j)} \varphi \tilde{R}_b^{(j+1)} \varphi Q_j^{(j+1)} \\ Q_r^{(j)} = R_b^{(j)} Q_t^{(j)} + T_t^{(j)} \varphi R_b^{(j+1)} \varphi Q_j^{(j+1)} \end{cases}$$

$$\begin{cases} \hat{\tilde{T}}_b^{(j)} = \left(\hat{I} - R_t^{(j)} \hat{\varphi} \hat{\tilde{R}}_b^{(j+1)} \hat{\varphi} \right)^{-1} \hat{T}_b^{(j)} \\ \hat{\tilde{R}}_b^{(j)} = \left(\tilde{R}_b^{j} + T_t^{j} \hat{\varphi} \hat{\tilde{R}}_b^{(j+1)} \hat{\varphi} \hat{\tilde{T}}_b^{j} \right) \end{cases}$$

While iterating, store T_b^j & R_b^j matrices

(4) Calculate $\vec{Q}_t^i, \vec{Q}_r^i$

Calculate transmittance/reflectance based on Poynting flux calculations!

Alternative formulation PBG→s

Note that boundary conditions (2.6) can be written in a different way. Introducing

(2.10)
$$\begin{bmatrix} 0 & k_{z2}^{(i)} \epsilon_2^{(i)} & 0 & -k_z^{(i)} \epsilon_2^{(i)} \\ 1 & 0 & 1 & 0 \\ -\frac{k_{z1}^{(i)}}{\mu^{(i)}\omega} & 0 & \frac{k_{z1}^{(i)}}{\mu^{(i)}\omega} & 0 \\ 0 & \frac{\omega \epsilon_1^{(i)} \epsilon_2^{(i)}}{c^2} & 0 & \frac{\omega}{c^2} \epsilon_1^{(i)} \epsilon_2^{(i)} \end{bmatrix} = N^{(i)} \quad \begin{bmatrix} E_x \\ E_y \\ H_x \\ H_y \end{bmatrix}$$

&

$$\hat{J}^{(i)} = \begin{bmatrix} e^{ik^{(i)}_{z,1} z_0} & 0 & 0 & 0 \\ 0 & e^{ik^{(i)}_{z,2} z_0} & 0 & 0 \\ 0 & 0 & e^{-ik^{(i)}_{z,1} z_0} & 0 \\ 0 & 0 & 0 & e^{-ik^{(i)}_{z,2} z_0} \end{bmatrix},$$

eqn. (2.6) transform to:

(2.11)
$$\hat{N}^{(i+1)} \hat{J}^{(i+1)} \begin{bmatrix} \vec{a}_t^{(i+1)} \\ \vec{a}_r^{(i+1)} \end{bmatrix} = \hat{N}^{(i)} \hat{J}^{(i)} \begin{bmatrix} \vec{a}_t^{(i)} \\ \vec{a}_r^{(i)} \end{bmatrix},$$

leading to:

(2.12)
$$\begin{bmatrix} \vec{a}_t^{(i+1)} \\ \vec{a}_r^{(i+1)} \end{bmatrix} = \underbrace{\hat{J}^{(i+1)^{-1}} \hat{N}^{(i+1)^{-1}} \hat{N}^{(i)} \hat{J}^{(i)}}_{T^{(i+1,i)}} \begin{bmatrix} \vec{a}_t^{(i)} \\ \vec{a}_r^{(i)} \end{bmatrix}$$

Now, the iteration becomes "trivial":

(2.12b)
$$\begin{bmatrix} \vec{a}_t^{(N+1)} \\ \tilde{\vec{a}}_i^{(N+1)} \end{bmatrix} = \underbrace{\left[\prod_{j=1}^{N} \hat{T}^{(j+1,j)} \right]}_{\hat{T}^{(N+1,1)}} \begin{bmatrix} a_i^{(1)} \\ a_r^{(1)} \end{bmatrix}$$

Now, we can go back to solving T/R problem:

(2.13)
$$\begin{bmatrix} \vec{a}_t^{(N+1)} \\ 0 \end{bmatrix} = \overbrace{\begin{bmatrix} \hat{T}_{11} & \hat{T}_{12} \\ \hat{T}_{21} & \hat{T}_{22} \end{bmatrix}}^{\hat{T}^{(N+1,1)}} \begin{bmatrix} \vec{a}_i^{(1)} \\ \vec{a}_r^{(1)} \end{bmatrix}$$

(2.13b)
$$\begin{cases} \hat{\tilde{R}} = -\hat{T}_{22}^{-1}\,\hat{T}_{21} \\ \hat{\tilde{T}} = \hat{T}_{11} + \hat{T}_{12}\,\hat{\tilde{R}} = \hat{T}_{11} - \hat{T}_{12}\,\hat{T}_{22}^{-1}\,\hat{T}_{21} \end{cases}$$

However, this implementation is often numerically unstable

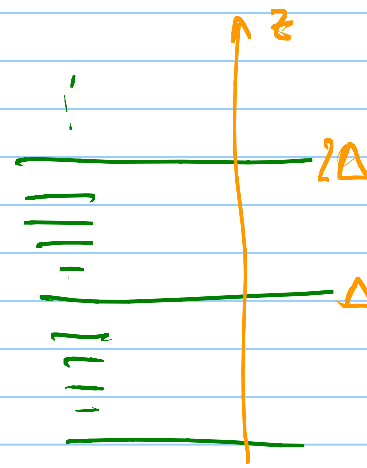
Nevertheless, the T-matrix formalism is beneficial for calculations of

(2.14)
$$\begin{bmatrix} \vec{a}_t^{(N+1)} \\ \\ 0 \end{bmatrix} = \hat{T}^{(N+1,1)} \begin{bmatrix} 0 \\ \\ \vec{a}_r^{(1)} \end{bmatrix}$$

=> dispersion can be given by:

(2.14b)
$$\hat{T}_{22}\,\vec{a}_r^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

→ photonic crystals:



Consider the periodic layered structure. According to the Bloch hypothesis, the field in such structure should be periodic as well!

$$(E, H)(\Delta) = e^{iq\Delta}(E, H)(0)$$

The field @ a given layer (ℓ) is given by:

$$\underbrace{\left[\begin{array}{c|c} \; & \; \\ \hline \; & \; \end{array}\right]}_{N^{(\ell)}\,\mathcal{J}^{(\ell)}} \begin{bmatrix} \vec{a}_t^{(\ell)} \\ \\ \vec{a}_r^{(\ell)} \end{bmatrix}$$

$\Rightarrow$ dispersion eq:

$$N^{(p+1)}\,\mathcal{J}^{(p+1)} \begin{bmatrix} \vec{a}_t^{(p+1)} \\ \vec{a}_r^{(p+1)} \end{bmatrix} = e^{iq\Delta}\, N^{(1)}\, \overset{I}{\cancel{\mathcal{J}^{(1)}(0)}} \begin{bmatrix} a_t^{(1)} \\ a_r^{(1)} \end{bmatrix}$$

$$\Rightarrow \left[\hat{N}^{(p+1)}\,\mathcal{J}^{(p+1)}(\Delta)\,\hat{T}^{(p+1,1)} - e^{iq\Delta}\, N^{(1)}\right] \begin{bmatrix} a_t^{(1)} \\ a_r^{(1)} \end{bmatrix} = 0$$

same medium

$\neq$

$$\hat{N}^{(1)} \underbrace{\left[\hat{\mathcal{J}}^{(p+1)}(\Delta)\,\hat{T}^{(p+1,1)} - e^{iq\Delta}\right]}_{\text{eigenvalue-like problem.}} \begin{bmatrix} \vec{a}_t^{(1)} \\ \vec{a}_r^{(1)} \end{bmatrix} = 0 \tag{2.15}$$

dispersion eq:

$$\det\left[\hat{\mathcal{J}}^{(p+1)}(\Delta)\,\hat{T}^{(p+1,1)} - e^{iq\Delta}\,\underline{\hat{I}}\right] = 0 \tag{2.15b}$$

in. both (2.14) & (2.15) disp. eq. gives $k_x(\omega)$ $[q(\omega, k_x)]$; while $\vec{a}$ give field distribution