# EM II - lecture 01 - plane waves

Note Title 12/14/2015

Start with Maxwell equations:

(1.1)
$$\begin{cases} \vec{\nabla}\times\vec{H} = \dfrac{\partial \vec{D}}{\partial t} + \vec{J} \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \end{cases}$$

with material equations:

(1.2)
$$\begin{cases} \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \varepsilon \vec{E}\,;\ \vec{P} = \varepsilon_0 \chi \vec{E} \\ \vec{H} = \dfrac{1}{\mu_0}\vec{B} - \vec{M}\,;\ \vec{B} = \mu_0 \mu \vec{H} \end{cases}$$

The rate of energy loss in the material is given by:

$\int \vec{J}\cdot\vec{E}\, d^3x$ ; From eq. (1.1):

(1.3)
$$\int \vec{J}\cdot\vec{E}\, d^3x = \int d^3x \left[(\vec{\nabla}\times\vec{H})\cdot\vec{E} - \frac{\partial \vec{D}}{\partial t}\cdot\vec{E}\right]$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - (\vec{\nabla}\times\vec{H})\cdot\vec{E} =$$

$$= -\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} - (\vec{\nabla}\times\vec{H})\cdot\vec{E}$$

$$\Rightarrow \int \vec{J}\cdot\vec{E}\, d^3x = -\int d^3x \left[\nabla\cdot(\vec{E}\times\vec{H}) + \frac{\partial \vec{D}}{\partial t}\cdot\vec{E} + \vec{H}\cdot\frac{\partial \vec{B}}{\partial t}\right]$$

$$\frac{\partial}{\partial t}(\vec{D}\cdot\vec{E}) = \vec{D}\cdot\frac{\partial \vec{E}}{\partial t} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} \simeq 2\,\vec{E}\cdot\frac{\partial \vec{D}}{\partial t}$$

in the limit $\epsilon \simeq$ const

$$\Rightarrow \vec{\nabla}\cdot\vec{S} + \frac{\partial u}{\partial t} = -\vec{J}\cdot\vec{E}\,,$$

(1.4) with
$$\begin{cases} u = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}) - \text{energy density} \\ \vec{S} = \vec{E}\times\vec{H} - \text{Poynting flux} = \text{energy flow} \end{cases}$$

Note $\vec{S}\perp\vec{E}$, $\vec{S}\perp\vec{H}$

Starting from (1.1) [in the source-free media with $\varepsilon \neq \varepsilon(\bar{r})$, $\mu \neq \mu(r)$, homogeneous media]

$$\bar{\nabla} \times \bar{\nabla} \times \bar{H} = \frac{\partial}{\partial t} \bar{\nabla} \times D = \frac{\partial}{\partial t} \varepsilon_0 \varepsilon \bar{\nabla} \times \bar{E} = -\varepsilon_0 \varepsilon \frac{\partial^2 B}{\partial t^2}$$

$$\bar{\nabla} \times \bar{\nabla} \times \bar{H} = \vec{\nabla} \cdot \cancel{(\operatorname{div} \vec{H})}^{0} - \Delta \vec{H} = -\varepsilon_0 \varepsilon \mu_0 \mu \frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\boxed{\Delta \vec{H} - \frac{\varepsilon \mu}{c^2} \frac{\partial^2}{\partial t^2} \vec{H} = 0}$$

wave eqn

(1.5) Alternatively,

$$\bar{\nabla} \times \bar{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \bar{\nabla} \times \bar{B} = -\mu_0 \mu \varepsilon_0 \varepsilon \frac{\partial^2 \bar{E}}{\partial t^2} \Rightarrow$$

$$\boxed{\left(\Delta - \frac{\varepsilon \mu}{c^2} \frac{\partial^2}{\partial t^2}\right) \vec{E} = 0}$$

$\frac{1}{v^2}$, where

$v = \frac{c}{n} = \frac{c}{\sqrt{\varepsilon \mu}}$ - velocity of light in matter

1D scalar wave eqn has general solution of

$$f(x - ct) + g(x + ct)$$

In 3D solution can be formed using plane-wave or other basis.

Eqn (1.1) ordinarily assume that all fields are real-valued. Mathematically it is more convenient to use complex exponentials and take the real part of the solution in the end.

Consider a solution of the form:

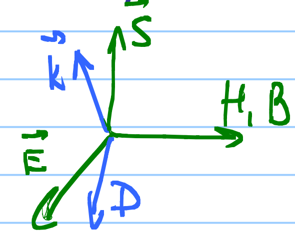
(1.6) $\vec{E} = \vec{E}_0 e^{i\bar{k}\cdot\bar{r} - i\omega t}$ ; $\bar{H} = \bar{H}_0 e^{i\bar{k}\cdot\bar{r} - i\omega t}$ , ...

(plane wave [[illegible]])

Substituting (1.6) => (1.1), obtain:

$$(1.7)\quad \begin{cases} i\vec{k}\times\vec{H}_0 = -i\omega\epsilon\epsilon_0\vec{E}_0 & \Rightarrow \vec{D}\perp\vec{k},\ \vec{D}\perp\vec{H} \\ i\vec{k}\times\vec{E}_0 = i\omega\mu_0\mu\vec{H}_0 & \Rightarrow \vec{B}\perp\vec{k},\ \vec{B}\perp\vec{E} \end{cases}$$

Note that in general, $\mu$, $\varepsilon$ may be tensors, which means that $\vec{D}$ may be not parallel to $\vec{E}$, in which case there is an angle between $\vec{S}$ & $\vec{k}$ ($\vec{E}, \vec{H}$ do not have to be $\perp\vec{k}$)




Similar situation arises when $\hat{\mu}$, $\vec{H} \not\parallel \vec{B}$

$$(1.7) \Rightarrow \vec{k}\times\vec{k}\times\vec{E}_0 = \vec{k}\cdot(\vec{k}\cdot\vec{E}_0) - \vec{E}_0 k^2 = \omega\mu_0\vec{k}\times\mu\vec{H}_0$$

Assuming isotropic $\mu$:

$$(1.8)\quad \vec{k}\cdot(\vec{k}\cdot\vec{E}_0) - \vec{E}_0 k^2 = -\frac{\omega^2}{c^2}\mu\epsilon\vec{E}_0$$

when $\epsilon$ is a scalar, $\vec{E}\parallel\vec{D}\perp\vec{k}$,

$$(1.9)\quad \boxed{\left(k^2 - \frac{\omega^2}{c^2}\mu\epsilon\right)\vec{E}_0 = 0} \Rightarrow$$

$\epsilon\mu$ must be $> 0$ for propagation

dispersion relation $\quad k^2 = \frac{\omega^2}{c^2}\epsilon\mu$,

$\vec{E}_0$ - any constant vector

The plane wave is identified by the wave-vector

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}, \quad k^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}\epsilon\mu$$

and by amplitude $(\vec{E}_0)$

Once $\vec{k}$ and $\vec{E}_0$ are calculated, the remaining components of the plane wave can be determined:

$$(1.10a)\quad \begin{cases} \vec{B}_0 = \dfrac{\vec{k}}{\omega} \times \vec{E}_0 = \hat{\mu}\mu_0 \vec{H}_0 \\ \vec{D}_0 = -\dfrac{\vec{k}}{\omega} \times \vec{H}_0 = \hat{\epsilon}\varepsilon_0 \vec{E}_0 \end{cases}$$

In the systems considered so far (and in majority of optical systems, $\mu$ is isotropic, moreover, $\mu = 1$)

We will assume $\mu = 1$ below, the case of $\mu \neq 1$ can be considered similarly to what is described below.

$$\rightarrow (1.10b)\quad \begin{cases} \vec{B}_0 = \dfrac{\vec{k}}{\omega} \times \vec{E}_0 \\ \vec{D}_0 = \varepsilon_0 \epsilon \vec{E}_0 \\ \vec{H}_0 = \dfrac{1}{\mu\mu_0} B_0 = \dfrac{1}{\mu_0} \dfrac{\vec{k}}{\omega} \times \vec{E}_0 \end{cases}$$

The energy flux is given by:

$$(1.11)\quad \vec{S} = \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right] = \frac{1}{2}\mathrm{Re}\left[\vec{E}_0 \times \vec{H}_0^*\right] =$$
$$= \frac{1}{2\mu_0\omega}\mathrm{Re}\left[\vec{E}_0 \times \vec{k}^* \times \vec{E}_0^*\right] = \frac{1}{2\mu_0\omega}\mathrm{Re}\left[\vec{k}^* |E|^2 - \vec{E}_0^*\left(\vec{k}^* \cdot \vec{E}_0\right)\right]$$

Note that real $\vec{E}_0$ yields linearly-polarized light, complex $E_0$ yields elliptical polarization

(remember, Real$[\vec{E}]$ represents solution]

it is possible to form a "helical" basis,

(1.12) $$\hat{e}^{\pm} = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$$

Note: $$\hat{e}^{\pm *} \cdot \hat{e}_z = 0$$

$$\hat{e}^{\pm *} \cdot \hat{e}^{\mp} = \frac{1}{2}(\hat{x}\cdot\hat{x} + i^2\hat{y}\cdot\hat{y}) = 0$$

$$\hat{e}^{\pm *} \cdot \hat{e}^{\pm} = 1$$

Consider a plane wave, with amplitude

(1.13) $$\vec{E}_0 = a_1 e^{i\delta_1}\hat{x} + a_2 e^{i\delta_2}\hat{y} = $$

$$= a_+ e^{i\delta_+}\hat{e}^+ + a_- e^{i\delta_-}\hat{e}^- =$$

$$= a_+ e^{i\delta_+}\frac{\hat{x} + i\hat{y}}{\sqrt{2}} + a_- e^{i\delta_-}\frac{\hat{x} - i\hat{y}}{\sqrt{2}} \Rightarrow$$

$$\begin{cases} a_1 e^{i\delta_1} = \frac{1}{\sqrt{2}}\left(a_+ e^{i\delta_+} + a_- e^{i\delta_-}\right) \\ a_2 e^{i\delta_2} = \frac{i}{\sqrt{2}}\left(a_+ e^{i\delta_+} - a_- e^{i\delta_-}\right) \end{cases}$$

$$\vec{E}_0^* = a_1 e^{-i\delta_1}\hat{x} + a_2 e^{-i\delta_2}\hat{y} =$$

$$= a_+ e^{-i\delta_+}\left(\frac{\hat{x} - i\hat{y}}{\sqrt{2}}\right) + a_- e^{-i\delta_-}\left(\frac{\hat{x} + i\hat{y}}{\sqrt{2}}\right)$$

$$\begin{cases} a_1 e^{-i\delta_1} = \frac{1}{\sqrt{2}}\left(a_+ e^{-i\delta_+} + a_- e^{-i\delta_-}\right) \\ a_2 e^{-i\delta_2} = \frac{-i}{\sqrt{2}}\left(a_+ e^{-i\delta_+} - a_- e^{+i\delta_-}\right) \end{cases}$$

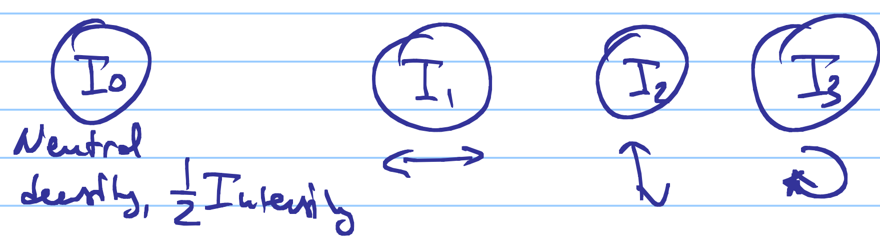
(1.14)
$$a_1^2 = |a_1|^2 = \frac{1}{2}\left(a_+^2 + a_-^2\right) + a_+ a_- \cos(\delta_- - \delta_+)$$
$$a_2^2 = |a_2|^2 = \frac{1}{2}\left(a_+^2 + a_-^2\right) - a_+ a_- \cos(\delta_- - \delta_+)$$

From measurement prospective, a plane wave is characterized by a set of Stokes param, obtained by measuring light passing through polarizers:

(1.15)

| linear polarization: | circ. polariz: |
| --- | --- |
| $S_0 = \lvert\hat{x}\cdot\vec{E}\rvert^2 + \lvert\hat{y}\cdot\vec{E}\rvert^2 = a_1^2 + a_2^2$ | $\lvert\hat{\epsilon}_+^*\cdot\vec{E}\rvert^2 + \lvert\hat{\epsilon}_-^*\cdot\vec{E}\rvert^2 = a_+^2 + a_-^2$ |
| $S_1 = \lvert\hat{x}\cdot\vec{E}\rvert^2 - \lvert\hat{y}\cdot\vec{E}\rvert^2 = a_1^2 - a_2^2$ | $2\mathrm{Re}\left[(\hat{\epsilon}^{+}\cdot\vec{E})^*(\hat{\epsilon}^{-}\cdot\vec{E})\right] = 2a_+a_-\cos(\delta^+ - \delta^-)$ |
| $S_2 = 2\mathrm{Re}\left[(\hat{x}\cdot\vec{E})^*(\hat{y}\cdot\vec{E})\right] = 2a_1a_2\cos(\delta_2 - \delta_1)$ | $2\mathrm{Im}\left((\hat{\epsilon}^{+}\cdot\vec{E})^*(\hat{\epsilon}^{-}\cdot\vec{E})\right) = 2a_+a_-\sin(\delta^+ - \delta^-)$ |
| $S_3 = 2\mathrm{Im}\left[(\hat{x}\cdot\vec{E})^*(\hat{y}\cdot\vec{E})\right] = 2a_1a_2\sin(\delta_2 - \delta_1)$ | $\lvert\hat{\epsilon}_+\cdot\vec{E}\rvert^2 - \lvert\hat{\epsilon}_-\cdot\vec{E}\rvert^2 = a_+^2 - a_-^2$ |

To measure Stokes parameters: take a set of four polarizers:




Then:

(1.16)
$$S_0 = 2I_0 \quad \leftarrow \text{relative intensity}$$
$$S_1 = 2I_1 - 2I_0 \quad \leftarrow \text{predominance of } \hat{x} \text{ axis}$$
$$S_2 = 2I_2 - 2I_0 \quad \leftarrow \text{predominance of "+"}$$
$$S_3 = 2I_3 - 2I_0 \quad \leftarrow \text{predominance of "−"}$$

Note: $S_0^2 \geq S_1^2 + S_2^2 + S_3^2$ - for time-averaged measurements.

# Anisotropic and gyrotropic media

In isotropic media the solutions were degenerate with respect to polarization of the wave. In anisotropic and gyrotropic materials permittivity is a tensor

Eq. (1.8) becomes:

(1.17) $$\vec{k}\cdot(\vec{k}\cdot\vec{E}_0) - \vec{E}_0 k^2 = -\frac{\omega^2}{c^2}\hat{\epsilon}\vec{E}_0$$

$$\begin{bmatrix} k_x^2-k^2 & k_x k_y & k_x k_z \\ k_x k_y & k_y^2-k^2 & k_y k_z \\ k_x k_z & k_z k_y & k_z^2-k^2 \end{bmatrix}\vec{E}_0 = -\frac{\omega^2}{c^2}\hat{\epsilon}\,\vec{E}_0$$

In the most simple case,

(1.18) $$\hat{\epsilon} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix}$$

Assuming that $k_y = 0$, $\Rightarrow \vec{k} = [k_x, 0, k_z]$,

(1.19) $$\begin{bmatrix} \frac{\omega^2}{c^2}\epsilon_1 - k_z^2 & 0 & k_x k_z \\ 0 & \frac{\omega^2}{c^2}\epsilon_1 - k_x^2 - k_z^2 & 0 \\ k_x k_z & 0 & \frac{\omega^2}{c^2}\epsilon_2 - k_x^2 \end{bmatrix}\begin{bmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{bmatrix} = 0$$

often, $\vec{n} = \frac{c}{\omega}\vec{k}$ is used.

then dispersion of the plane waves is given by:

$$(1.19a)\quad \det\begin{vmatrix} \epsilon_1 - n_z^2 & 0 & n_x n_z \\ 0 & \epsilon_1 - n_x^2 - n_z^2 & 0 \\ n_x n_z & 0 & \epsilon_2 - n_x^2 \end{vmatrix} = 0$$

$$(\epsilon_1 - n_z^2)(\epsilon_1 - n_x^2 - n_z^2)(\epsilon_2 - n_x^2) - n_x^2 n_z^2 (\epsilon_1 - n_x^2 - n_z^2) = 0$$

(1.20a)

ⓐ $\epsilon_1 - n_x^2 - n_z^2 = 0 \Leftrightarrow k_x^2 + k_z^2 = \epsilon_1 \frac{\omega^3}{c^2}$

$\vec{E}_0 = \begin{bmatrix} 0 \\ E_0 \\ 0 \end{bmatrix}$ - ordinary wave

ⓑ $(\epsilon_1 - n_z^2)(\epsilon_2 - n_x^2) - n_x^2 n_z^2 = 0$

$\epsilon_1 \epsilon_2 - \epsilon_1 n_x^2 - \epsilon_2 n_z^2 = 0$

$$\frac{n_x^2}{\epsilon_2} + \frac{n_z^2}{\epsilon_1} = 1 \Leftrightarrow \frac{k_x^2}{\epsilon_2} + \frac{k_z^2}{\epsilon_1} = \frac{\omega^2}{c^2}$$

(1.20b)

$(\epsilon_1 - n_z^2) E_x + n_x n_z E_z = 0$

$\left(\epsilon_1 - \epsilon_1 + \frac{n_x^2 \epsilon_1}{\epsilon_2}\right) E_x + n_x n_z E_z = 0 \Rightarrow E_z n_z \epsilon_2 = - n_x \epsilon_1 E_x$

$$\vec{E}_0 = \begin{bmatrix} n_z \epsilon_2 \\ 0 \\ -n_x \epsilon_1 \end{bmatrix} E_0 \Leftrightarrow \begin{bmatrix} 1 \\ \\ -\frac{n_x \epsilon_1}{n_z \epsilon_2} \end{bmatrix} E_0$$