Vectors in Cartesian Coordinate Systems

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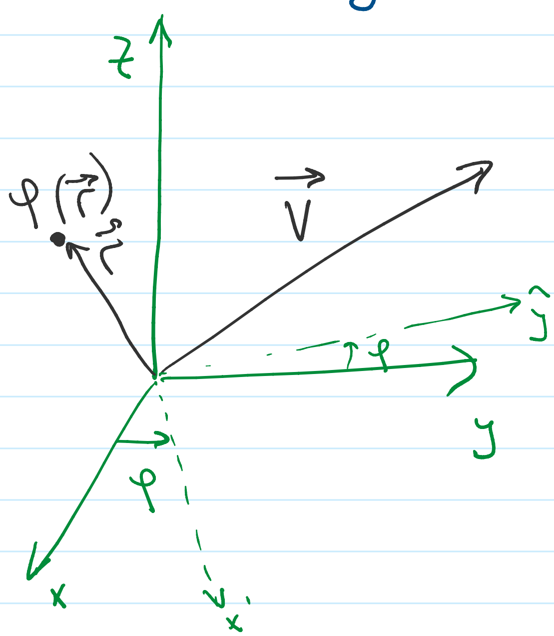
Vector :

→ An object with n elements

→ An object with magnitude and direction

→ An object with n elements that transform in a certain way



Note that the vector is different from scalar, $\varphi(\vec{x})$

Indeed, the value of scalar functions does not change under coordinate transformations:

$$\varphi(\vec{x}) = \varphi'(\vec{x}\,')$$

(1.1) in Cartesian coordinates : $\vec{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$

(1.1) in Cartesian coordinates: $V = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$

$$= V_x' \hat{x}' + V_y' \hat{y}' + V_z' \hat{z}'$$

Since $\hat{x}_i \cdot \hat{x}_j = \delta_{ij}$ and $\hat{x}_i' \cdot \hat{x}_j' = \delta_{ij}$

$$V_x' = \hat{x}' \cdot \vec{V} = V_x \hat{x} \cdot \hat{x}' + V_y \hat{y} \cdot \hat{x}' + V_z \hat{z} \cdot \hat{x}'$$

. . .

(1.2) $$\Rightarrow \boxed{V_i' = \sum \hat{x}_i' \cdot \hat{x}_j V_j}$$

When $\vec{V} = \vec{r} = \sum_i x_i \hat{x}_i$, it is seen that

$$\hat{x}_i = \frac{\partial \vec{r}}{\partial x_i} \quad \left[\text{ex: } \hat{y} = \frac{\partial}{\partial y}(x\hat{x} + y\hat{y} + z\hat{z}) \right]$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x'\hat{x}' + y'\hat{y}' + z'\hat{z}' \quad \left\{ \frac{\partial}{\partial y} \right.$$

$$\Rightarrow \hat{y} = \frac{\partial x'}{\partial y}\hat{x}' + \frac{\partial y'}{\partial y}\hat{y}' + \frac{\partial z'}{\partial y}\hat{z}'$$

. . .

$$\hat{x}_j = \sum_i \frac{\partial x_i'}{\partial x_j} \hat{x}_i'$$

$$\Rightarrow \hat{x}_j \cdot \hat{x}_i' = \frac{\partial x_i'}{\partial x_j}$$

(1.3) $$\Rightarrow \boxed{V_i' = \sum_j \frac{\partial x_i'}{\partial x_j} V_j}$$

$$\left[\text{in Cartesian coordinates } \frac{\partial x_i'}{\partial x_j} = \cos(x_i', x_j) \right]$$

Operations with vectors (fixed coord. system)

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$$a\vec{V} = \hat{x}(aV_x) + \hat{y}(aV_y) + \hat{z}(aV_z) \leftarrow \text{vector}$$

$$\vec{V}\cdot\vec{W} = V_xW_x + V_yW_y + V_zW_z \leftarrow \text{scalar} \quad \text{, in particular:}$$

$$|\vec{V}|^2 = \bar{V}\cdot\bar{V}$$

$$\bar{V}\times\bar{W} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix} = \hat{x}\left(V_yW_z - V_zW_y\right) - \hat{y}\left(V_xW_z - V_zW_x\right) + \hat{z}\left(V_xW_y - V_yW_x\right) \leftarrow \text{pseudo-vector}$$

(vector in rotations, "−" in reflection)

We can introduce differential operator:

$$\vec{\nabla} = \sum \hat{x}_i \frac{\partial}{\partial x_i} \quad ; \quad \left[\vec{\nabla}' = \sum \hat{x}_i' \frac{\partial}{\partial x_i'}\right]$$

Assuming that $\varphi(\vec{x})$ is some scalar function:

gradient $\vec{\nabla}\varphi$ is [possibly] a vector

$$\boxed{\left(\bar{\nabla}'\varphi\right)_i = \frac{\partial\varphi}{\partial x_i'} = \sum_j \frac{\partial\varphi}{\partial x_j}\frac{\partial x_j}{\partial x_i'} = \sum_j \frac{\partial x_j}{\partial x_i'}\left(\vec{\nabla}\varphi\right)_j} \tag{1.4}$$

Note that relationship (1.4) is similar, but not identical to (1.3). In cartesian coordinates,

$$\frac{\partial x_i'}{\partial x_j} = \frac{\partial x_j}{\partial x_i'} = \cos(x_i' x_j) \, ,$$

but it does not hold in curvilinear coordinates

divergence: $\vec{\nabla}\cdot\vec{V} = \sum \frac{\partial V_i}{\partial x_i}$ [scalar in Cart coord] divergence : $\vec{\nabla} \cdot \vec{V} = \sum_i \frac{\partial V_i}{\partial x_i}$ [scalar in Cart. coord)

curl : $\vec{\nabla} \times \vec{V}$ [pseudo-v in Cart. coord]

Vectors in curvilinear coordinates

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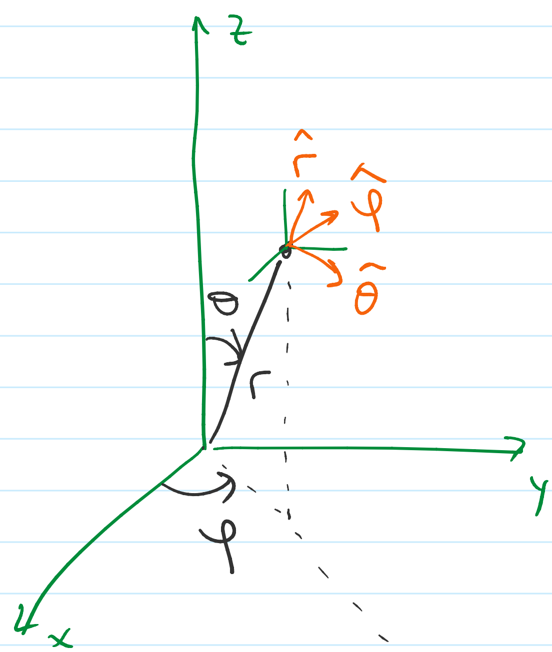
In Cartesian coordinates, the trio of unit (basis) vectors is fixed and does not depend on the point in space.

However, in many other systems this is not the case.

Consider, for example, spherical coordinates:



The location in space is given by the trio (r, θ, φ) [note the order!].

Note:

→ $\hat{r}, \hat{\theta}, \hat{\varphi}$ are still orthogonal to each other

→ the direction of the unit vectors depends on the coordinates

To properly define the basis (unit) vectors, we follow the procedure inspired by Cartesian coordinates:

To properly define the basis (unit) vectors, we follow the procedure inspired by Cartesian coordinates:

Assume that

$$\vec{r} = \sum q_i \vec{q}_i \text{ , then:}$$

basis vectors: $\vec{q}_i = \frac{\partial \vec{r}}{\partial q_i}$ [Note: these are not neccesarily unit vectors]:

$$d\vec{r} = \sum_i dq_i \vec{q}_i = \sum_i \frac{\partial \vec{r}}{\partial q_i} dq_i$$

scalar: $$(dl)^2 = \sum_i \frac{\partial \vec{r}}{\partial q_i} dq_i \sum_j \frac{\partial \vec{r}}{\partial q_j} dq_j = \sum_{i,j} \underbrace{\frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j}} dq_i dq_j =$$

$$= \sum_{i,j} g_{ij} \, dq_i \, dq_j$$

g_{ij} ↓ metric tensor

In general: $|\vec{V}|^2 = \sum_{i,j} g_{ij} V_i V_j$

Example: metric tensor of spherical coordinates

$q_1 = r, \; q_2 = \theta, \; q_3 = \varphi;$

$$\vec{r} = r\hat{r} = \underbrace{x\hat{x} + y\hat{y} + z\hat{z}} = r\cos\varphi\sin\theta\,\hat{x} + r\sin\varphi\sin\theta\,\hat{y} + r\cos\theta\,\hat{z}$$

depends on r, θ, φ => use Cartesian

$$g_{11} = \left(\frac{\partial \vec{r}}{\partial r}\right) \cdot \left(\frac{\partial \vec{r}}{\partial r}\right) = \left[\hat{x}\cos\varphi\sin\theta + \hat{y}\sin\varphi\sin\theta + \hat{z}\cos\theta\right]$$

$$\cdot \left[\hat{x}\cos\varphi\sin\theta + \hat{y}\sin\varphi\sin\theta + \hat{z}\cos\theta\right] =$$

$$= \cos^2\varphi\sin^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\theta = 1$$

$$\ldots \cos^2\varphi \sin^2\theta + \sin^2\varphi \sin^2\theta + \cos^2\theta = 1$$

$$g_{12} = \left(\frac{\partial \vec{r}}{\partial r}\right)\cdot\left(\frac{\partial \vec{r}}{\partial \theta}\right) = \left[\hat{x} \cos\varphi \sin\theta + \hat{y} \sin\varphi \sin\theta + \hat{z}\cos\theta\right]$$

$$\cdot\left[\hat{x}\, r\cos\varphi\cos\theta + \hat{y}\, r\sin\varphi\cos\theta - \hat{z}\, r\sin\theta\right]$$

$$= r\cos\theta\sin\theta\left[\cos^2\varphi + \sin^2\varphi - 1\right] = 0$$

$$g_{22} = \left(\frac{\partial \vec{r}}{\partial \theta}\right)\cdot\left(\frac{\partial \vec{r}}{\partial \theta}\right) = r^2\cos^2\varphi\cos^2\theta + r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\theta = r^2$$

$$g_{33} = \left(\frac{\partial \vec{r}}{\partial \varphi}\right)\cdot\left(\frac{\partial \vec{r}}{\partial \varphi}\right) = \left[-\hat{x}\, r\sin\varphi\sin\theta + \hat{y}\, r\cos\varphi\sin\theta\right]\cdot$$

$$\left[-\hat{x}\, r\sin\varphi\sin\theta + \hat{y}\, r\cos\varphi\sin\theta\right] = r^2\sin^2\theta$$

all other $g_{ij} = 0$

Note that g_{ij} is represented by a diagonal matrix

$$g_{ij} = h_i^2\,\delta_{ij},$$

where h_i are called scaling factors.

The fact that g_{ij} is diagonal reflects orthogonality of the coordinate system.

In orthogonal systems (even in curvilinear) it is possible to introduce unit vectors:

$$|\vec{q}_i|^2 = \sum_j g_{ij}\, q_i\, q_j = g_{ii}\, q_i^2 = h_i^2$$

↑ has only i-th component, which is equal to 1

$$\hat{q}_i = \frac{1}{h_i}\frac{\partial \vec{r}}{\partial q_i}\,; \quad |\hat{q}_i|^2 = 1$$

Note that direction of unit vectors changes from one point to the next.

For, example:

$$h_i \hat{q}_i = \frac{\partial \vec{r}}{\partial q_i} \quad \Big\{ \frac{\partial}{\partial q_j}, \quad j \neq i$$

$$\frac{\partial h_i}{\partial q_j} \hat{q}_i + h_i \frac{\partial \hat{q}_i}{\partial q_j} = \frac{\partial^2 \vec{r}}{\partial q_i \partial q_j} = \frac{\partial h_j}{\partial q_i} \hat{q}_j + h_j \frac{\partial \hat{q}_j}{\partial q_i}$$

Since all $\hat{q}_i$ are orthogonal, $\frac{\partial \hat{q}_i}{\partial q_j} \perp \hat{q}_i$

$$\Rightarrow \boxed{\frac{\partial \hat{q}_i}{\partial q_j} = \frac{1}{h_i} \frac{\partial h_j}{\partial q_i} \hat{q}_j} \quad , \ i \neq j$$

$$\hat{q}_i \cdot \hat{q}_j = \delta_{ij} \quad \Big\{ \frac{\partial}{\partial q_j}$$

$$-\frac{1}{h_i}\frac{\partial h_j}{\partial q_i}$$

$$\frac{\partial \hat{q}_i}{\partial q_j} \cdot \hat{q}_j + \frac{\partial \hat{q}_j}{\partial q_j} \cdot \hat{q}_i = 0 \Rightarrow \underbrace{\frac{\partial \hat{q}_j}{\partial q_j} \cdot \hat{q}_i}_{i\text{-th comp. of } \frac{\partial \hat{q}_j}{\partial q_j}} = - \overbrace{\frac{\partial \hat{q}_i}{\partial q_j} \cdot \hat{q}_j}$$

$$\Rightarrow \frac{\partial \hat{q}_j}{\partial q_j} = -\sum_{i \neq j} \frac{1}{h_i} \frac{\partial h_j}{\partial q_i} \hat{q}_i$$

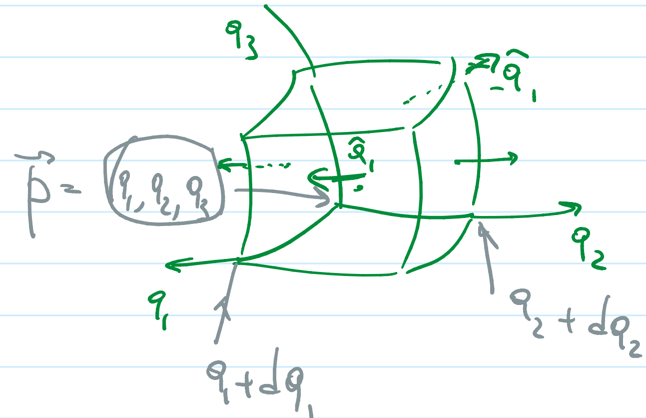
Differential operators:

$$\vec{\nabla}\varphi = \lim_{\int d\tau \to 0} \frac{\oint \varphi \, d\vec{a}}{\int d\tau}$$

$$\vec{\nabla}\cdot\vec{V} = \lim_{\int d\tau \to 0} \frac{\oint d\vec{a}\cdot\vec{V}}{\int d\tau}$$

$$\vec{\nabla}\times\vec{V} = \lim_{\int d\tau \to 0} \frac{\oint d\vec{a}\times\vec{V}}{\int d\tau}$$

Example: gradient in curvilinear coordinates



$$\oint \varphi\, d\vec{a} = \int \overbrace{h_2 h_3\, dq_2\, dq_3}^{d^2a}\, \varphi\, \hat{q}_1 \Big|_{\vec{p}+dq_1} - h_2 h_3\, dq_2\, dq_3\, \varphi\, \hat{q}_1 \Big|_{\vec{p}} + h_1\, dq_1\, h_3\, dq_3\, \hat{q}_2\, \varphi \Big|_{\vec{p}+dq_2} - h_1\, dq_1\, h_3\, dq_3\, \hat{q}_2\, \varphi \Big|_{\vec{p}}$$

$$+ \int h_1 h_2\, dq_1\, dq_2\, \varphi\, \hat{q}_3 \Big|_{p+dq_3} - h_1 h_2\, dq_1\, dq_2\, \varphi\, \hat{q}_3 \Big|_{p}$$

$$= \int dq_2\, dq_3 \Big[\cancel{h_2 h_3 \varphi \hat{q}_1 \Big|_p} + \frac{\partial}{\partial q_1}(h_2 h_3 \varphi \hat{q}_1)\Big|_p dq_1 - \cancel{h_2 h_3 \varphi \hat{q}_1 \Big|_p}$$

$$+ \Big| \frac{\partial}{} (h\, h\, \varphi\, \hat{q})\Big| + \frac{\partial}{} (h\, h\, \varphi\, \hat{q})\Big| \Big]\, dq\, dq\, dq$$

$$+\left[\frac{\partial}{\partial q_2}\left(h_1 h_3 \varphi \hat{q}_2\right)\Big|_p + \frac{\partial}{\partial q_3}\left(h_1 h_2 \varphi \hat{q}_3\right)\Big|_p\right] dq_1\, dq_2\, dq_3$$

$$= \Big[\cancel{\frac{\partial h_2}{\partial q_1} h_3 \varphi \hat{q}_1} + \cancel{h_2 \frac{\partial h_3}{\partial q_1} \varphi \hat{q}_1} + h_2 h_3 \frac{\partial \varphi}{\partial q_1}\hat{q}_1 + h_2 h_3 \varphi \frac{\partial \hat{q}_1}{\partial q_1}$$

$$- h_2 h_3 \varphi\left(\cancel{\frac{1}{h_2}\frac{\partial h_1}{\partial q_2}\hat{q}_2} + \cancel{\frac{1}{h_3}\frac{\partial h_1}{\partial q_3}\hat{q}_3}\right)$$

$$+ \cancel{\frac{\partial h_1}{\partial q_2} h_3 \varphi \hat{q}_2} + \cancel{h_1 \frac{\partial h_3}{\partial q_2}\varphi \hat{q}_2} + h_1 h_3 \frac{\partial \varphi}{\partial q_2}\hat{q}_2 + h_1 h_3 \varphi \frac{\partial \hat{q}_2}{\partial q_2}$$

$$- h_1 h_3 \varphi\left(\cancel{\frac{1}{h_1}\frac{\partial h_2}{\partial q_1}\hat{q}_1} + \cancel{\frac{1}{h_3}\frac{\partial h_2}{\partial q_3}\hat{q}_3}\right)$$

$$+ \cancel{\frac{\partial h_1}{\partial q_3} h_2 \varphi \hat{q}_3} + \cancel{h_1 \frac{\partial h_2}{\partial q_3}\varphi \hat{q}_3} + h_1 h_2 \frac{\partial \varphi}{\partial q_3}\hat{q}_3 + h_1 h_2 \varphi \frac{\partial \hat{q}_3}{\partial q_3}\Big]$$

$$- h_1 h_2 \varphi\left(\cancel{\frac{1}{h_1}\frac{\partial h_3}{\partial q_1}\hat{q}_1} + \cancel{\frac{1}{h_2}\frac{\partial h_3}{\partial q_2}\hat{q}_2}\right)$$

$$\times dq_1\, dq_2\, dq_3$$

$$\Rightarrow \vec{\nabla}\varphi = \lim_{\dots} \frac{dq_1\, dq_2\, dq_3 \left(h_2 h_3 \frac{\partial \varphi}{\partial q_1}\hat{q}_1 + h_1 h_3 \frac{\partial \varphi}{\partial q_2}\hat{q}_2 + h_1 h_2 \frac{\partial \varphi}{\partial q_3}\hat{q}_3\right)}{h_1 h_2 h_3\, dq_1\, dq_2\, dq_3}$$

$$\vec{\nabla}\varphi = \sum_i \frac{1}{h_i}\frac{\partial\varphi}{\partial q_i}\hat{q}_i$$

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(h_2 h_3 V_1\right) + \frac{\partial}{\partial q_2}\left(h_1 h_3 V_2\right) + \frac{\partial}{\partial q_3}\left(h_1 h_2 V_3\right)\right]$$

$$\vec{\nabla}\times\vec{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{q}_1 & h_2\hat{q}_2 & h_3\hat{q}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

Covariance, vectors, and tensors

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We already saw that under coordinate transformation some objects do not change, some change in linear manner, which could be either

$$V_i' = \sum \frac{\partial q_i'}{\partial q_j} V_j$$

or

$$V_i' = \sum \frac{\partial q_j}{\partial q_i'} V_j$$

The first kind of objects we called scalars, while the second kind - vectors. (components of vectors, to be precise)

To distinguish different kind of transformations, we introduce contravariant $[V^\alpha]$ and covariant $[V_\alpha]$ components.

By convention: coordinates will be assumed contravariant:

$$q^i$$

$\Rightarrow$ Basis vectors are 'covariant': $\vec{e}_i = \frac{\partial \vec{r}}{\partial q^i}$

$$\Rightarrow \quad \underbrace{\vec{V}}_{\text{"true" vector}} = \underbrace{\overbrace{V^i}^{\text{Contravariant components}} \vec{e}_i}_{\text{implied summation } \left(\sum_{i=1}^{D}\right)}$$

transformation rules: $V'^i = \frac{\partial q'^i}{\partial q^j} V^j$

Metric tensor: $g_{ij} = \vec{e}_i \cdot \vec{e}_j$ [2-nd rank covariant tensor]

transformation rules: $A'_{ij} = \frac{\partial q^k}{\partial q'^i} \frac{\partial q^l}{\partial q'^j} A_{kl}$

Metric tensor can be used to "lower" indices (transform contravariant components into covariant)

$$\vec{V} = V_i \vec{e}^{\,i}, \quad \text{where} \quad \boxed{V_i = g_{ij} V^j}$$

One can define a contravariant metric tensor:

$$g^{ij} \Leftarrow g^{ij} g_{jk} = \delta^i_k$$

where δ^i_k - mixed 2-nd rank (invariant) tensor:

$$\delta'^i_k = \frac{\partial q'^i}{\partial q^l} \frac{\partial q^m}{\partial q'^k} \underbrace{\delta^l_m}_{\begin{cases} 0, & l \neq m \\ 1, & l = m \end{cases}} = \frac{\partial q'^i}{\partial q^m} \frac{\partial q^m}{\partial q'^k} = \frac{\partial q'^i}{\partial q'^k} = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}$$

g^{ik} can be used to "raise" indices ($\vec{\varepsilon}^{\,i} = g^{ik} \vec{\varepsilon}_k$)

In general, tensor of rank N has N indices and is defined via transformation law.

Quotient theorem: if tensor expression is valid in all coord. systems and all but one quantities are tensors, the remaining object is valid in all coord. systems and all but one quantities are tensors, the remaining object is also a tensor

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Covariant differential operators

$$\vec{\nabla}\varphi = \frac{\partial \varphi}{\partial q^i}\,\vec{\mathcal{E}}^i$$

Note: covariant component

$$\frac{\partial \vec{V}}{\partial q^i} = \frac{\partial}{\partial q^i} V^k \vec{\mathcal{E}}_k = \frac{\partial V^k}{\partial q^i}\vec{\mathcal{E}}_k + V^k \frac{\partial \vec{\mathcal{E}}_k}{\partial q^i}$$

Introducing $\frac{\partial \vec{\mathcal{E}}_k}{\partial q^i} = \Gamma^m_{ik}\vec{\mathcal{E}}_m$

not a tensor

$$= \frac{\partial V^k}{\partial q^i}\vec{\mathcal{E}}_k + V^k \Gamma^m_{ik}\vec{\mathcal{E}}_m = \frac{\partial V^k}{\partial q^i}\vec{\mathcal{E}}_k + V^m \Gamma^k_{im}\vec{\mathcal{E}}_k =$$

$$= \underbrace{\left(\frac{\partial V^k}{\partial q^i} + \Gamma^k_{im} V^m\right)}\vec{\mathcal{E}}_k$$

contravariant component of vector
⇒ transforms like a 2nd rank tensor

$$V^k{}_{;i} = \frac{\partial V^k}{\partial q^i} + \Gamma^k_{im} V^m$$

$$\vec{\nabla}\cdot\vec{V} = V^i{}_{;i}$$