

Displacement current

Thursday, September 22, 2016 3:25 PM

Up until now the equations governing EM response look like:

$$(7.1) \left\{ \begin{array}{l} \text{div } \vec{E} = \frac{\rho}{\epsilon_0} \\ \text{div } \vec{B} = 0 \\ \text{curl } \vec{H} = \vec{j} \end{array} \right. \Rightarrow \vec{\nabla} \cdot \vec{j} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0$$
$$\text{curl } \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

However, moving charge represents current,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \Rightarrow \text{the equations (7.1)}$$

are not compatible with electrodynamics.

To make them compatible, Maxwell introduced displacement current

$$\vec{j}_d = \frac{\partial \vec{D}}{\partial t} = \frac{\partial}{\partial t} \epsilon_0 \epsilon \vec{E} = \frac{\partial}{\partial t} \epsilon_0 \vec{E} + \frac{\partial \vec{P}}{\partial t}$$

full Maxwell eqs become:

$$(7.1a) \left\{ \begin{array}{l} \text{div } \vec{E} = \frac{\rho}{\epsilon_0} \\ \text{div } \vec{B} = 0 \end{array} \right.$$

(7.1a) $\left. \begin{aligned} \text{div } \vec{B} &= 0 \\ \text{curl } \vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{j} \\ \text{curl } \vec{E} &= - \frac{\partial \vec{B}}{\partial t} \end{aligned} \right\} \Rightarrow \vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0 = \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{D} + \vec{\nabla} \cdot \vec{j} = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j}$

As before, we can introduce potentials V, \vec{A}
 where $\vec{B} = \vec{\nabla} \times \vec{A}$ $\left[\Leftarrow \text{div } \vec{B} = \text{div } \text{curl } \vec{A} = 0 \right]$

now $\text{curl } \vec{E} = - \frac{\partial}{\partial t} (\text{curl } \vec{A}) \Rightarrow$

$$\text{curl} \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = - \vec{\nabla} V$$

$$\vec{E} = - \vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

Maxwell eqs in potentials (in vacuum) becomes

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \epsilon_0 \frac{\partial}{\partial t} \left(- \vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \right) + \vec{j}$$

$$\vec{\nabla} \cdot \left(\vec{\nabla} V + \frac{\partial \vec{A}}{\partial t} \right) = - \frac{\rho}{\epsilon_0}$$

(other eqs are automatically satisfied)

$$\vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \Delta \vec{A} = - \epsilon_0 \vec{\nabla} \frac{\partial V}{\partial t} - \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} + \vec{j}$$

$$\nabla^2 V + \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = - \frac{\rho}{\epsilon_0}$$

(7.2)

$$\nabla^2 \vec{A} - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right) - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = - \mu_0 \vec{j}$$

Note that the equations don't change under gauge transformations:

$$(7.3) \quad \boxed{\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda; \quad V \rightarrow V - \frac{\partial\Lambda}{\partial t}}, \text{ where}$$

Λ - arbitrary function.

$$(7.4) \quad \Rightarrow \text{Lorentz gauge: } \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0$$

$$\Rightarrow \text{M. eq: } \left\{ \begin{array}{l} \nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j} \end{array} \right. \left. \begin{array}{l} \text{two wave eqn} \\ \text{"instantaneous" Coulomb solution} \end{array} \right.$$

Coulomb gauge:

$$\vec{\nabla} \cdot \vec{A} = 0$$

$$(7.5) \quad \left\{ \begin{array}{l} \nabla^2 \phi = -\rho/\epsilon_0 \Rightarrow \phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d^3r' \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j} + \frac{1}{c^2} \vec{\nabla} \frac{\partial \phi}{\partial t} \end{array} \right. \left. \begin{array}{l} \text{inhomogeneous wave eqn} \\ -\mu_0 \vec{j}_{\text{eff}} \end{array} \right.$$

Any vector can be written as a sum of

"longitudinal" (irrotational) vector: $\vec{\nabla} \times \vec{J}_\ell = 0$

"transverse" (solenoidal) vector: $\vec{\nabla} \cdot \vec{J}_t = 0$

Eq. (7.5c) becomes:

$$(7.6) \quad \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j}_t$$

$$(7.6) \left\{ \begin{array}{l} \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = \mu_0 j_t \\ \vec{\nabla} \frac{\partial \phi}{\partial t} = \frac{1}{\epsilon_0} \frac{\partial \vec{j}_e}{\partial t} \end{array} \right.$$

Note that when no sources are present,

$$(7.7) \left\{ \begin{array}{l} \phi = 0 ; \nabla^2 \vec{A} = \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} ; \\ \vec{E} = - \frac{\partial \vec{A}}{\partial t} ; \vec{B} = \vec{\nabla} \times \vec{A} \end{array} \right.$$

Green function for wave equation

Monday, October 10, 2016 11:43 AM

Consider wave equations:

$$(7.7) \quad \nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -4\pi f(\vec{r}, t)$$

in free space (no boundary surfaces);

In Fourier domain

$$(7.8) \quad \int \psi(\vec{r}, t) = \frac{1}{2\pi} \int \psi(\vec{r}, \omega) e^{-i\omega t} d\omega$$

$$\psi(\vec{r}, \omega) = \int \psi(\vec{r}, t) e^{i\omega t} dt$$

⇒ Eq. (7.7) becomes:

$$(7.9) \quad \nabla^2 \psi(\vec{r}, \omega) + \frac{\omega^2}{c^2} \psi(\vec{r}, \omega) = -4\pi f(\vec{r}, \omega)$$

The latter eq. can be solved using Green's function formalism,

GR eqn:

$$(7.10) \quad (\nabla^2 + k^2) G_k(\vec{r}, \vec{r}') = -4\pi \delta(\vec{r} - \vec{r}')$$

from symmetry consideration $G_k(\vec{r}, \vec{r}') = G_k(\vec{R})$,

where $\vec{R} = \vec{r} - \vec{r}'$

$$(7.10a) \quad \Rightarrow \frac{1}{R} \frac{d^2}{dR^2} (R G_k) + k^2 G_k = -4\pi \delta(\vec{R})$$

$$\textcircled{1} R \neq 0: \quad \frac{1}{R} \frac{d^2}{dR^2} (R G_k) + k^2 G_k = 0$$

$$\Rightarrow \frac{d^2}{dr^2} (rG_k) = -k^2 (rG_k) \Rightarrow$$

$$G_k = \frac{1}{r} (A e^{ikr} + B e^{-ikr}) = A G_k^+(r) + B G_k^-(r)$$

(7.11)

$$G_k^\pm = \frac{e^{\pm ikr}}{r}$$

In the limit $r \rightarrow 0$: $\frac{1}{r} \frac{d^2}{dr^2} (rG_k) \approx -4\pi \delta(r) \Rightarrow G \sim \frac{1}{r}$

To figure out A, B constants, look at time dependence:

$$G^\pm(r, t-t') = \frac{1}{2\pi} \int G_k^\pm(r, \omega) e^{-i\omega(t-t')} d\omega$$

$$\begin{aligned} \left(\nabla_r^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G^\pm &= \left(\nabla_r^2 + \frac{\omega^2}{c^2} \right) G(r, \omega) \frac{e^{-i\omega(t-t')}}{2\pi} = \\ &= -4\pi \delta(r) \int \frac{e^{-i\omega(t-t')}}{2\pi} d\omega = -4\pi \delta(r-r') \delta(t-t') \end{aligned}$$

$\Rightarrow G^\pm(r, t-t')$ satisfies G.F. for wave eqn. we can calculate $G(r, t)$ explicitly:

$$\begin{aligned} (7.12) \quad G^+(r, t) &= \frac{1}{2\pi} \int \frac{1}{r} e^{+ikr - i\omega(t-t')} d\omega \\ &= \frac{1}{2\pi r} \int e^{-i\omega(t-t' + \frac{r}{c})} d\omega = \frac{1}{r} \delta\left(t + \frac{r}{c} - t'\right) \end{aligned}$$

$G^+(r, t) \rightarrow$ retarded Green's function (causal)

$G^-(R, \tau) \rightarrow$ advanced G . F (used when $\psi(r, t \rightarrow \infty)$ known)

\Rightarrow solution to wave eqn:

$$(7.13) \quad \psi(\vec{r}, t) = \psi_h(\vec{r}, t) + \iint f(\vec{r}', t') G^+(\vec{r} - \vec{r}', t - t') d^3\vec{r}' dt' =$$

$$= \psi_h(\vec{r}, t) + \int \frac{f(\vec{r}', t')}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$

$t' = t - \frac{|\vec{r} - \vec{r}'|}{c}$

Poynting theorem

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$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

The total work done by the fields:

$$(7.14) \quad \int \vec{J} \cdot \vec{E} d^3r = \int (\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t}) \cdot \vec{E} d^3r = \\ = \int [\vec{E} \cdot (\vec{\nabla} \times \vec{H}) - \frac{\partial \vec{D}}{\partial t} \cdot \vec{E}] d^3r$$

$$(7.15) \quad \vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) \\ \Rightarrow \int \vec{J} \cdot \vec{E} d^3r = - \int \left[\vec{H} \cdot \left(\frac{\partial \vec{B}}{\partial t} \right) + \vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] d^3r$$

$\frac{\partial u}{\partial t}, u = \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H})$

$$(7.16) \Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\vec{J} \cdot \vec{E}}, \leftarrow \text{conservation law}$$

$$(7.16a) \quad \boxed{\vec{S} = \vec{E} \times \vec{H}} \quad \text{Poynting vector; energy flux}$$

In the presence of losses,

$$\vec{E}(\vec{r}, t) = \frac{1}{2\pi} \int \vec{E}(\vec{r}, \omega) e^{-i\omega t} d\omega$$

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \int d\omega \int d\omega' \frac{1}{4\pi^2} \vec{E}(\vec{r}, \omega) \epsilon(\omega) (-i\omega) \vec{E}(\vec{r}, \omega') e^{-i(\omega+\omega')t}$$

$\dots \int d\omega' \frac{1}{4\pi^2} \vec{E}(\vec{r}, \omega') \epsilon(\omega') (-i\omega') \vec{E}(\vec{r}, \omega) e^{-i(\omega-\omega')t}$

dispersion to be discussed later

$$\begin{aligned}
&= \text{Re} \left[\int \int d\omega d\omega' \frac{1}{4\pi^2} \vec{E}_i(r, -\omega') (-i\omega \epsilon(\omega)) \vec{E}_i(r, \omega) e^{-i(\omega - \omega')t} \right] = \\
&= \frac{1}{2} \int \int \frac{d\omega d\omega'}{4\pi^2} \vec{E}_i^*(r, -\omega') \underbrace{(i\omega' \epsilon^*(\omega') - i\omega \epsilon(\omega))}_{\substack{\text{when } \omega' \approx \omega \\ \rightarrow 2\omega \text{Im}(\epsilon) - i(\omega - \omega') \frac{d}{d\omega}(\omega \epsilon^*(\omega)) + \dots}} \vec{E}_i(r, \omega) e^{-i(\omega - \omega')t}
\end{aligned}$$

$$\begin{aligned}
(7.17) \quad & \left\langle \vec{E}_i \cdot \frac{\partial \vec{D}}{\partial t} \right\rangle = \omega_0 \text{Im}(\epsilon_0) \langle E_i^2 \rangle + \frac{\partial u_{\text{eff}}^{(e)}}{\partial t} \\
& u_{\text{eff}}^{(e)} = \frac{1}{2} \text{Re} \left[\frac{d(\omega \epsilon)}{d\omega} \right] \langle E_i^2 \rangle
\end{aligned}$$