Multipole Expansion

Wednesday, August 31, 2016 10:00 AM

Assume that we have some charge distribution

The potential due to this distribution is given by:

(4.1) $$V(\vec{r}) = \int \frac{\rho(\vec{r}')}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|} d^3r' = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\vec{r}') \cdot \frac{1}{|\vec{r}-\vec{r}'|} =$$

$$= \frac{1}{4\pi\epsilon_0} \int \rho(\vec{r}') d^3r' \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi) =$$

for potential outside localized charge distribution:

$r_< = r'$; $r_> = r$

$$= \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \underbrace{\int \rho(\vec{r}') r'^l Y^*_{lm}(\theta', \varphi') d^3r'}_{q_{lm} \text{ - multipole moment}}$$

(4.2)
$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}}$$

$$q_{lm} = \int \rho(r') r'^l Y^*_{lm}(\theta', \varphi') d^3r'$$

Note: @ $r \to \infty$, the potential is dominated by the lowest-l non-vanishing multipoles

For example:

$$q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(\vec{r}') d^3r' = \frac{q}{\sqrt{4\pi}} \quad \text{(monopole)}$$

(4.2a) [illegible]

(42a) $$q_{11} = -\sqrt{\frac{3}{8\pi}} \int (x'-iy')\rho(\bar r')d^3r' = -\sqrt{\frac{3}{8\pi}}(p_x - ip_y)$$
$$q_{10} = \sqrt{\frac{3}{4\pi}} \int z'\rho(\bar r')d^3r' = -\sqrt{\frac{3}{4\pi}}\, p_z$$ (dipole)

...

where: $q = \int \rho(\bar r')d^3r'$ - total charge

$\vec p = \int \vec r'\rho(\bar r')d^3r'$ - dipole moment, ...

In general:

$$q_{l,-m} = (-1)^m q^*_{l,m}$$

Alternatively, Taylor expansion can be used to calculate potential at large distances

(43) $$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\bar r')}{|r-r'|}d^3\bar r' \simeq \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\vec r')}{|r-r'|} + \rho(r')\vec r'\cdot\nabla'\left(\frac{1}{|r-r'|}\right)+\dots\right]d^3r'$$

$$\frac{1}{\sqrt{(\alpha-\alpha')^2+a^2}} \underset{\alpha'\ll\alpha,a}{\simeq} \frac{1}{\sqrt{\alpha^2+a^2}} + \frac{\alpha}{(\alpha^2+a^2)^{3/2}}\alpha' + \left[3\frac{\alpha^2}{(\alpha^2+a^2)^{5/2}} - \frac{1}{(\alpha^2+a^2)^{3/2}}\right]\frac{\alpha'^2}{2} + \dots =$$

$$= \frac{1}{\sqrt{\alpha^2+a^2}} + \frac{\alpha}{(\alpha^2+a^2)^{3/2}}\alpha' + \frac{2\alpha^2-a^2}{(\alpha^2+a^2)^{5/2}}\frac{\alpha'^2}{2}$$

$$\frac{1}{|\bar r - \bar r'|} = \frac{1}{\sqrt{\dots}} \simeq$$

$$\sqrt{(x-x')^2+(y-y')^2+(z-z')}$$

$$\frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$+\frac{xx'+yy'+zz'}{r^3}+\frac{1}{2r^5}\Big\{x'^2(3x^2-r^2)+y'^2(3y^2-r^2)$$

$$+z'^2(3z^2-r^2)+3x'y'xy+3x'z'xz+3y'z'yz\Big\}=$$

$$=\dots\ \frac{1}{2r^5}\left(3+\dots-x'^2x^2-x'^2y^2-x'^2z^2-y'^2x^2-y'^2y^2-y'^2z^2-z'^2z^2-z'^2x^2-z'^2y^2\right)$$

$$=\dots\ \frac{1}{2r^5}\left(3+\dots-r'^2(x^2+y^2+z^2)\right)$$

$$=\frac{1}{r}+\frac{\bar{r}\cdot\bar{r}'}{r^3}+\frac{1}{2}\sum_{i,j}\frac{(3x_i'x_j'-\delta_{ij}r'^2)}{r^5}x_ix_j$$

(4.3) $$\Rightarrow V(\bar{r})\simeq\frac{1}{4\pi\epsilon_0}\left[\frac{q}{r}+\frac{\bar{p}\cdot\bar{r}}{r^3}+\frac{1}{2}\sum_{i,j}Q_{ij}\frac{x_ix_j}{r^5}+\dots\right],$$

$$q=\int\rho(\bar{r}')\,d^3r' \quad \text{- charge}$$

$$\vec{p}=\int\vec{r}'\rho(\bar{r}')\,d^3r' \quad \text{- dipole moment}$$

$$Q_{ij}=\int(x_i'x_j'-\delta_{ij}r'^3)\rho(\bar{r}')\,d^3r' \quad \text{- quadrupole moment}$$

Note that the first non-vanishing moment is independent of origin, values of higher-order moments depend on origin

Finally, we can calculate the field due to dipole moment:

$$E_x = -\frac{1}{4\pi\epsilon_0}\frac{\partial}{\partial x}\frac{p_x x + p_y y + p_z z}{r^3} = -\frac{p_x}{r^3} + 3\frac{(\bar{p}\cdot\bar{r})}{r^5}x$$

(1.4)
$$\vec{E} = -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\bar{p}\cdot\bar{r})\vec{r} - \vec{p}\,r^2}{r^5} - \frac{4\pi}{3}\vec{p}\,\delta(\bar{r})\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{3(\bar{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3} - \frac{4\pi}{3}\vec{p}\,\delta(\bar{r})\right]$$

from, e.g. averaging of field, see textbook

Consider a polarizable media. The total charge inside can be represented as a sum of

→ free (bulk) charges and

→ bound (polarization) charges

(4.5) $$\begin{cases} \rho(r') = \rho_{free} + \sum_i N_i \langle e_i \rangle \\ \text{with} \quad \sum_i N_i \langle e_i \rangle = 0 \end{cases}$$

i-th type of molecule

⇒ the contribution of bound charges yields dipole moment:

$$\vec{P}(\vec{r}\,') = \sum_i N_i \langle p_i \rangle \; ; \; Q_{lm} = \ldots = 0$$

Now, the potential due to total charge (see 4.3)

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho_f(r')}{|r-r'|} + \vec{P}(r') \vec{\nabla}' \left(\frac{1}{|r-r'|} \right) \right] d^3 r' =$$

$$= \frac{1}{4\pi\epsilon_0} \int \frac{\rho_f(\bar{r}') - \vec{\nabla}' \cdot \vec{P}(\bar{r}')}{|\bar{r}-\bar{r}'|} d^3 \bar{r}' =$$

(4.6) $$= \frac{1}{4\pi\epsilon_0} \int \frac{\tilde{\rho}(\bar{r}')}{|\bar{r}-\bar{r}'|} d^3 \bar{r}'$$

where $\tilde{\rho}(\bar{r}') = \rho_f - \vec{\nabla}' \cdot \vec{P}(r')$

(4.7) $$\Rightarrow \quad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \tilde{\rho} = \frac{1}{\epsilon_0} (\rho_f - \vec{\nabla} \cdot \vec{P})$$

(4.7) $\Rightarrow \quad \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\tilde{\rho} = \frac{1}{\epsilon_0}\left(\rho_f - \vec{\nabla}\cdot\vec{P}\right)$

introduce $\quad \vec{D} = \epsilon_0\vec{E} + \vec{P}$ (electric displacement)

(4.8) $\vec{\nabla}\cdot\vec{D} = \rho_f$

the second equation

$\vec{\nabla}\times\vec{E} = 0$

is not affected by the polarization charge

In linear isotropic dielectrics

(4.9) $\vec{D} = \epsilon\vec{E}$

In linear anisotropic dielectrics

$\vec{D} = \hat{\epsilon}\vec{E}$;

in coordinate form:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Note: $\vec{D}$ may be not parallel to $\vec{E}$

Eqs. (4.9) yield boundary conditions!

(4.10) $\begin{cases} E_\tau = \text{const} \\ \Delta D_n = \sigma_f \end{cases} \quad \left[\ \Delta E_n = \epsilon_0\sigma_f + \sigma_p\ \right]$

Note that inside isotropic homogeneous uniform dielectrics

$$\vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot\epsilon\vec{E} = \epsilon\,\vec{\nabla}\cdot\vec{E} = \rho_f$$

(4.11) $$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\epsilon}}$$

$\Rightarrow$ The solution using Laplace / Poisson eqn depending on free charges with subsequent rescaling is appropriate

Energy of Charge Distribution and in Materials

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(4.11)
$$\boxed{W} = \int \rho(\vec r)\, V(\vec r)\, d^3r \simeq$$

$$\int \rho(\vec r)\left[V(0) + \vec r\cdot\vec\nabla V\Big|_0 + \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial^2 V}{\partial x_i \partial x_j}\Big|_0 + \dots\right] d^3r$$

$\cancel{+\frac{r^3}{3}} + \frac{1}{6} r^2\, \vec\nabla\cdot\vec E \to 0$ for external field

$$= \int \left[\rho(\vec r) V(0) - \rho(\vec r)\,\vec r\cdot\vec E - \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial E_j}{\partial x_i} + \dots\right]$$

$$\boxed{= q V(0) - \vec p\cdot\vec E(0) - \frac{1}{6}\sum_{ij} Q_{ij} \frac{\partial E_j}{\partial x_i}\Big|_0 + \dots}$$

for two point dipoles

(4.12)
$$W_{12} = -\vec p_1\cdot\vec E_2 = -\vec p_1 \cdot \frac{3(\hat r\cdot\vec p_2)\hat r - \vec p_2}{4\pi\epsilon_0 r^3} = \frac{\vec p_1\cdot\vec p_2 - 3(\hat r\cdot\vec p_1)(\hat r\cdot\vec p_2)}{4\pi\epsilon_0 r^3}$$

In dielectrics, $\underbrace{\text{the energy to assemble}}$ the system:

(4.13)
$$W = \frac{1}{2}\int \rho(\vec r)\, V(\vec r)\, d^3r$$

$$\delta W = \int \delta\rho(\vec r)\cdot V(\vec r)\, d^3r = \int \delta(\vec\nabla\cdot\vec D)\cdot V(\vec r)\, d^3r =$$

$$= \int \vec\nabla\cdot(\delta\vec D)\cdot V(\vec r)\, d^3r = -\int \delta\vec D\cdot(\vec\nabla V)\, d^3r =$$

$$= \int \vec{E}\cdot\delta\vec{D}\, d^3r$$

in linear media : $\vec{E}\cdot(\delta\vec{D}) = \frac{1}{2}\,\delta(\bar{E}\cdot\bar{D}) \Rightarrow$

(4.14) $$W = \frac{1}{2}\int \bar{E}\cdot\bar{D}\, d^3r$$

To illustrate the above relation, assume that a polarizable body is inserted into a region of space such as the external charges remain fixed

Then,

$$\Delta W = \frac{1}{2}\int\left[\bar{E}\cdot\bar{D} - \vec{E}_0\cdot\vec{D}_0\right]d^3r$$

$$(\bar{E}+\bar{E}_0)(\bar{D}-\bar{D}_0) = \bar{E}\cdot\bar{D} + \bar{E}_0\bar{D} - \bar{E}\cdot\bar{D}_0 - \bar{E}_0\cdot\bar{D}_0$$

$$= \frac{1}{2}\int(\bar{E}\cdot\bar{D}_0 - \bar{E}_0\cdot\bar{D})\,d^3r + \underbrace{\frac{1}{2}\int(\bar{E}+\bar{E}_0)(\bar{D}-\bar{D}_0)\,d^3r}_{=0 \text{ (see book)}}$$

$$= \frac{1}{2}\int(\epsilon_0-\epsilon)\,\vec{E}\cdot\vec{E}_0 = -\frac{1}{2}\int\vec{P}\cdot\vec{E}_0\, d^3r$$

$\Rightarrow$ energy density of a dielectric in external field: with fixed sources

(4.15) $$w = -\frac{1}{2}\vec{P}\cdot\vec{E}_0$$

$\Rightarrow$ dielectric is pulled into region of higher field

(4.16) $$\vec{F} = -\left(\frac{\partial W}{\ldots}\right)$$

(4.16) $$\vec{F} = -\left(\frac{\partial W}{\partial z}\right)_Q$$

if potential, not external charge is fixed,

$$\delta W = \frac{1}{2}\int(\rho\,\delta V + V\,\delta\rho)\,d^3r =$$

$$\delta W = \underbrace{\frac{1}{2}\int\rho\,\delta V_1}_{\text{disconnect electrodes, move dielectrics}} + \underbrace{\frac{1}{2}\int\left[\rho(-\delta V_1) + V\,\delta\rho\right]d^3r}_{\text{connect electrodes, move charges to return potential}}$$

$$= -\frac{1}{2}\int\rho\,\delta V_1\,d^3r$$

(4.17) $$\Rightarrow \quad \delta W_V = -\delta W_Q$$

(4.18) $$\vec{F} = \left(\frac{\partial W}{\partial z}\right)_V$$

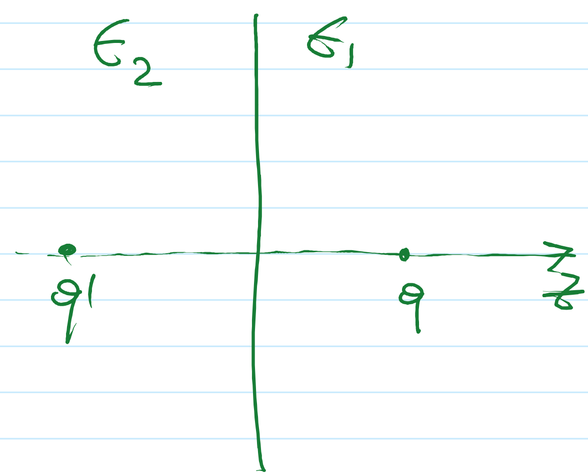
Similar to solving boundary-value problems in vacuum, electrostatic problems in dielectrics can be reduced to a combination of solutions of Laplace eqn and boundary conditions.

Since there are no free charges, we can use Laplace eqn for finding fields inside every homogeneous layer.

Method of images can be generalized for dielectric media.

Example:

find a field of a charge distance d away from the interface between two media



Solving eqns:

$$(4.19)\quad \begin{cases} \epsilon_1 \vec{\nabla} \cdot \vec{E} = q\delta(z-d), & z>0 \\ \epsilon_2 \vec{\nabla} \cdot \vec{E} = 0, & z<0 \\ \vec{\nabla} \times \vec{E} = 0 \Rightarrow & \vec{E} = -\vec{\nabla} V \end{cases}$$

$$(4.20)\quad \begin{cases} E_x\big|_{z=0} = \text{const} \\ E_y\big|_{z=0} = \text{const} \end{cases}$$

(4.20)
$$\begin{cases} E_y\big|_{z=0} = \text{const} \\ \epsilon_i E_z\big|_{z=0} = \text{const} \end{cases}$$

put image charge q', same distance as q from the interface, and a charge q'' @ the location of the charge q.

Remembering that we neglect "image" charges in their own region, obtain:

(4.21)
$$V = \frac{1}{4\pi} \begin{cases} \left[\frac{q}{|\vec{r}-d\hat{z}|} + \frac{q'}{|\vec{r}+d\hat{z}|}\right]\frac{1}{\epsilon_1}, & z>0 \\ \frac{q''}{|\vec{r}-d\hat{z}|}\frac{1}{\epsilon_2}, & z<0 \end{cases}$$

Now, applying boundary condition

$$E_x = \text{const}: \frac{(q+q')}{4\pi\epsilon_1}\frac{x}{(\sqrt{x^2+y^2})^3} = \frac{q''}{4\pi\epsilon_2}\frac{x}{(\sqrt{x^2+y^2})^3}$$

$$D_z = \text{const}: \quad -q+q' = -q''$$

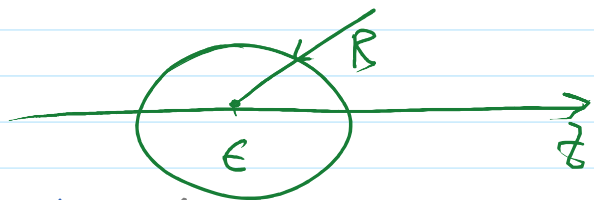
$$\Rightarrow \begin{cases} \frac{q+q'}{\epsilon_1} = \frac{q''}{\epsilon_2} \\ q-q' = q'' \end{cases} \Rightarrow \begin{cases} \frac{q+q'}{\epsilon_1} - \frac{q-q'}{\epsilon_2} = 0 \\ \frac{2q}{\epsilon_1} = q''\left[\frac{1}{\epsilon_2}+\frac{1}{\epsilon_1}\right] \end{cases}$$

$$q' = -q\,\frac{\epsilon_2-\epsilon_1}{}$$

(4.22)
$$\begin{cases} q' = -q \dfrac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} \\ q'' = q \dfrac{2\epsilon_2}{\epsilon_2+\epsilon_1} \end{cases}$$

Consider now a problem of a dielectric sphere in a constant field

Solution to Laplace eqn (no free charges)



(4.23)
$$\begin{cases} V_{out} = \sum \left[\alpha_\ell^+ r^\ell + \alpha_\ell^- \dfrac{1}{r^{\ell+1}}\right] P_\ell(\cos\theta) \\ V_{in} = \sum \beta_\ell^+ r^\ell P_\ell(\cos\theta) \end{cases}$$

[axially symm problem]

Note that for $r\to\infty$ $\alpha_1^+ = -1 \Leftarrow V \to -E_0 z = -\alpha_1^+ r\cos\theta$

all others $\alpha_\ell^+ = 0 \Rightarrow$ all other $\alpha_\ell^- = \beta_\ell^+ = 0$

Boundary conditions @ $r = R$

$$\begin{array}{ll} E_\tau : & \left\{ -\dfrac{1}{R}\dfrac{\partial V_{in}}{\partial\theta} = -\dfrac{1}{R}\dfrac{\partial V_{out}}{\partial\theta} \right. \\ \epsilon E_n : & \left. -\epsilon\dfrac{\partial V_{in}}{\partial r} = -\epsilon_0 \dfrac{\partial V_{out}}{\partial r} \right. \end{array}$$

$$\Big\{ \ \alpha_1^+ R + \frac{\alpha_1^-}{R^2} = \beta_1^+ R$$

$$\begin{cases} \alpha_1^+ R + \dfrac{\alpha_1^-}{R^2} = \beta_1^+ R \\[2ex] \epsilon_0 \left[\alpha_1^+ - \dfrac{2\alpha_1^+}{R^3} \right] = \epsilon \, \beta_1^+ \end{cases}$$

$$\beta_1^+ = \alpha_1^+ \frac{3\epsilon_0}{2\epsilon_0 + \epsilon} = -\frac{3}{2 + \epsilon/\epsilon_0} E_0$$

$$\alpha_1^- = -\alpha_1^+ \frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon} R^3 = \frac{\epsilon/\epsilon_0 - 1}{2 + \epsilon/\epsilon_0} R^3 E_0$$

$$(4.24) \quad \boxed{\vec{E}_{in} = \frac{3}{2 + \epsilon/\epsilon_0} \vec{E}_0 \; ; \; \vec{P}_{in} = (\epsilon - \epsilon_0) E_0 = 3\epsilon_0 \frac{\epsilon/\epsilon_0 - 1}{2 + \epsilon/\epsilon_0} \vec{E}_0}$$

$$V_{out} = -E_0 r \cos\theta + \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} \frac{E_0 R^3}{r^2} \cos\theta =$$

$$= -E_0 r \cos\theta + \frac{1}{4\pi\epsilon_0} \frac{\bar{p} \cdot \bar{r}}{r^3} ,$$

$$(4.25) \quad \boxed{\vec{p} = 4\pi\epsilon_0 \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} R^3 \vec{E}_0}$$

Note: $\vec{P}_{in} \cdot \frac{4\pi}{3} R^3 = \frac{4\pi R^3}{\cancel{3}} \cancel{3}\epsilon_0 \frac{\epsilon/\epsilon_0 - 1}{2 + \epsilon/\epsilon_0} \bar{E}_0 = 4\pi\epsilon_0 \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} R^3 \vec{E}_0$

$= \vec{p}$

Molecular Polarizability and Permittivity

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Dense materials can be considered as collection of large number of polarizable molecules. Polarization of each molecule is given by

(4.26) $$\vec{p} = \epsilon_0 \gamma_{mol} \left(\vec{E}_i + \vec{E}_0\right)$$

where $\vec{E}_i$ is the electric field due to other molecules and $\vec{E}_0$ is the macroscopic "excitation" field.

(4.27) $$E_i = E_{near} - E_p$$

where E_{near} is the "true" field due to nearby molecules and E_p is the field due to "averaged" polarization of the media

(4.28) $$E_{near} = \sum_{i,j,k} \frac{3(\vec{p}\cdot\vec{r}_{ijk})\vec{r}_{ijk} - r_{ijk}^2 \vec{p}}{r_{ijk}^5} = 0$$

For example, E^x_{near} for cubic lattice $(x_i = \Delta \cdot i,\ y_j = \Delta \cdot j,\ z_k = \Delta \cdot k)$

$$E^x_{near} = \sum_{i,j,k} \frac{\Delta i\left(3p_x \cdot \Delta \cdot i + 3p_y \cdot \Delta \cdot j + 3p_z \Delta \cdot k\right) - \Delta^2(i^2+j^2+k^2)p_x}{\Delta^5(i^2+j^2+k^2)^{5/2}} =$$

$$= \cancel{\sum_{i,j,k} \frac{3i^2 - i^2 - j^2 - k^2}{\Delta^3(i^2+j^2+k^2)^{5/2}}} p_x + \cancel{\sum_{i,j,k} \frac{3p_y ij + 3p_z ik}{\Delta^3(i^2+j^2+k^2)^{5/2}}} = 0$$

$$= \sum_{ijk} \frac{\cancel{\cdots}}{\Delta^3 (i^2+j^2+k^2)^{5/2}} p_x + \sum_{ijk} \frac{\cancel{\cdots}}{\Delta^3 (i^2+j^2+k^2)^{5/2}}$$

$0 \quad (\langle i^2\rangle = \langle j\rangle^2 = \langle k^2\rangle)$ $\qquad$ $0 \quad (\langle ij\rangle = \langle ik\rangle = 0)$

(4.28b) $\vec{E}_p = -\frac{\vec{P}}{3\epsilon_0}$, with $\vec{P} = N \cdot \vec{p} = \epsilon_0 \chi \vec{E}_0$

$$\Rightarrow \quad \vec{p} = \epsilon_0 \gamma_m \left(\frac{\vec{P}}{3\epsilon_0} + \vec{E}_0 \right) = \frac{\vec{P}}{N}$$

$$\epsilon_0 N \gamma_m \left(\frac{\chi}{3} + 1 \right) E_0 = \epsilon_0 \chi E_0$$

(4.29)

$$\boxed{\chi = \frac{N\gamma_m}{1 - \frac{N\gamma_m}{3}}} \quad \text{or}$$

$$\boxed{\gamma_m} = \frac{1}{N}\frac{\chi}{1+\chi/3} = \frac{3}{N}\frac{\chi}{\chi+3} = \boxed{\frac{3}{N}\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}}$$

$\Rightarrow$ the ratio $\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} \propto N$

Dielectric Permittivity of a Mixture; Metamaterials

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Permittivity of mixture can be related to composition and permittivity of components. Consider, for example material formed by array of dispersed spheres (permittivity ϵ_i) in the host (permittivity ϵ_h) in the dilute limit.

The field inside individual spheres

(4.30) $$\vec{E}_i = \frac{3}{\epsilon_i/\epsilon_h + 2}\vec{E}_h$$

$$\Rightarrow \epsilon_{eff} = \frac{\langle \epsilon E\rangle}{\langle E\rangle} = \frac{p\,\epsilon_i E_i + (1-p)E_h\epsilon_h}{p\,E_i + (1-p)E_h} =$$

$$= \frac{p\,\frac{3\epsilon_i}{\epsilon_i/\epsilon_h+2} + (1-p)\epsilon_h}{p\,\frac{3}{\epsilon_i/\epsilon_h+2} + (1-p)} =$$

$$\epsilon_{eff} = \frac{3p\epsilon_i + (1-p)(\epsilon_i+2\epsilon_h)}{3p\epsilon_h + (1-p)(\epsilon_i+2\epsilon_h)} =$$

(4.31) $$\boxed{\epsilon_{eff} = \frac{(1+2p)\epsilon_i + 2(1-p)\epsilon_h}{(2+p)\epsilon_h + (1-p)\epsilon_i}}$$

Note that resonance (known as Maxwell-Garnett resonance) is concentration-dependent, happens @

(4.32) $$\epsilon_i = -\frac{2+p}{1-p}\,\epsilon_h$$