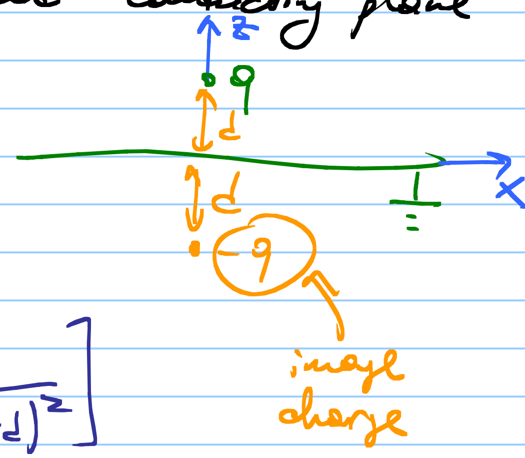
EM I. Method of images

Note Title 7/11/2016

Method of images is widely used to find potential in regions of space, surrounded by conductors by placing image charges in the "virtual" space behind the conductors to satisfy boundary conditions

① Charge near grounded conducting plane



(2.1) B.C. $\varphi(x, z=0) = 0$

Potential:

(2.2) $\varphi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+(z+d)^2}}\right]$

(B.C. works); applicable only for $z \geq 0$

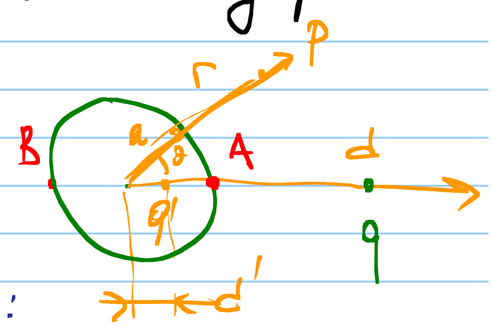
(2.3) $\sigma = -\varepsilon_0 \frac{\partial\varphi}{\partial\vec{n}}\cdot\hat{n} = -\varepsilon_0 \left.\frac{\partial\varphi}{\partial z}\right|_{z=0} =$

$= \frac{q}{4\pi}\left[\frac{-2d}{\sqrt{x^2+d^2}^{\,3}}\right] = -\frac{qd}{2\pi\left(\sqrt{x^2+d^2}\right)^3}$

(2.4) $F = -\frac{q^2}{4\pi\varepsilon_0 4d^2} = -\frac{q^2}{16\pi\varepsilon_0 d^2}$

② point charge near grounded conducting sphere

Due to cylindrical symmetry, it settles to consider only the plane of q, q', and center of the sphere



Look for a potential @ A, B:

(2.5) $\varphi(A) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{d-a} + \frac{q'}{a-d'}\right) = 0 = \varphi(B) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{d+a} + \frac{q'}{a+d'}\right]$

$$\Rightarrow \quad q' = \frac{q(d'-a)}{d-a}$$

$$\frac{d'+a}{d+a} + \frac{d'-a}{d-a} = 0$$

$$-(d'+a)(d-a) = (d'-a)(d+a)$$

$$-dd' + a^2 - \cancel{ad} + \cancel{ad'} = dd' - a^2 - \cancel{ad} + \cancel{ad'}$$

(2.6) $$d' = \frac{a^2}{d} \; ; \; q' = \frac{q}{d-a}\left(\frac{a^2}{d} - a\right) = -q\frac{a}{d}$$

$\Rightarrow$ potential:

(2.5a) $$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-d)^2+y^2}} - \frac{a}{d}\frac{1}{\sqrt{(x-d')^2+y^2}}\right]$$

Force:

(2.7) $$F = -\frac{q^2}{4\pi\epsilon_0 (d - a^2/d)^2} = -\frac{q^2}{4\pi\epsilon_0}\frac{d^2}{(d^2-a^2)^2}$$

induced surface charge density:

$$\sigma = \left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = \frac{\partial}{\partial r}\left\{\frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(r\cos\theta - d)^2 + (r\sin\theta)^2}} - \frac{a}{d}\cdots\right]\right\}$$

$$= \frac{q}{4\pi\epsilon_0}\left\{\frac{\cos\theta(a\cos\theta - d) + a\sin^2\theta}{\left((a\cos\theta - d)^2 + a^2\sin^2\theta\right)^{3/2}} - \frac{a}{d}\frac{\cos\theta(a\cos\theta - d') + a\sin^2\theta}{\left[(a\cos\theta - d')^2 + a^2\sin^2\theta\right]^{3/2}}\right\}$$

$$= \frac{q}{4\pi\epsilon_0}\left\{\frac{a - d\cos\theta}{(a^2+d^2-2ad\cos\theta)^{3/2}} - \frac{a}{d}\frac{a - d'\cos\theta}{(a^2+d'^2-2ad'\cos\theta)^{3/2}}\right\} =$$

$$= \frac{q}{4\pi\epsilon_0}\left\{\frac{a - d\cos\theta}{(a^2+d^2-2ad\cos\theta)^{3/2}} - \frac{a}{d}\underbrace{\frac{a - \frac{a^2}{d}\cos\theta}{\left(a^2 + \frac{a^4}{d^2} - 2\frac{a^3}{d}\cos\theta\right)^{3/2}}}\right\}$$

$$\frac{a}{d}\frac{\frac{a}{d}(d - a\cos\theta)}{\left[\frac{a^2}{d^2}(d^2+a^2-2ad\cos\theta)\right]^{3/2}} = \frac{d}{a}\frac{(d - a\cos\theta)}{(a^2+d^2-2ad\cos\theta)^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_0\left[a^2+d^2-2ad\cos\theta\right]^{3/2}}\left[a-\cancel{d\cos\theta}-\frac{d^2}{a}+\cancel{d\cos\theta}\right]=$$

$$= -\frac{q}{4\pi\epsilon_0}\,\frac{d^2\left[1-\frac{a^2}{d^2}\right]}{a d^3\left[1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta\right]^{3/2}}$$

(2.8) $$\Rightarrow \sigma = -\frac{q}{4\pi\epsilon_0}\,\frac{\left(1-a^2/d^2\right)}{ad\left(1+a^2/d^2-2a/d\cos\theta\right)^{3/2}}$$

Green's function:

(2.5b) $$\boxed{G(r,r_0)=\frac{1}{|\vec{r}-\vec{r}_0|}-\frac{a}{r_0}\,\frac{1}{\left|\vec{r}-a^2/r_0\,\hat{r}_0\right|}}$$

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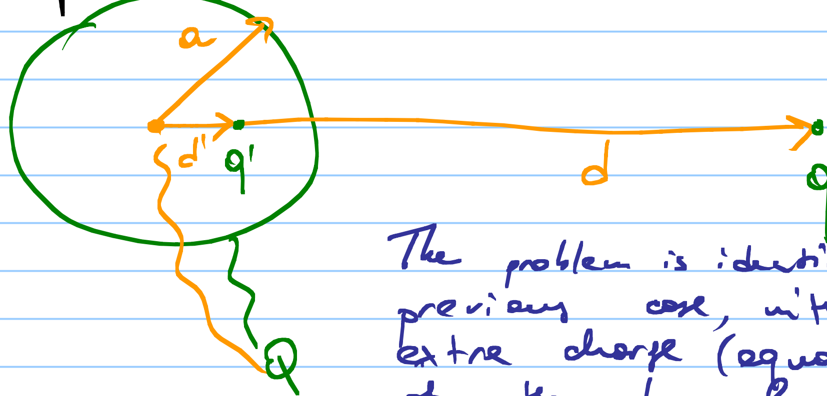
3 Point charge near charged conducting sphere.



The problem is identical to previous case, with one extra charge (equal to that of the sphere, located at the center)

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4 Conducting sphere in a homogeneous field

A uniform electric field can be considered as a limit of a field created by two point charges, $+Q$ & $-Q$, located @ $x=R$, $x=-R$, as $R\to\infty$

(2.9) Indeed, $$\varphi_0(x)=\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{|x-R|}-\frac{1}{|x+R|}\right]=$$

$$=\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{(R-x)}-\frac{1}{(R+x)}\right]\simeq\frac{Q}{4\pi\epsilon_0}\left[\frac{2x}{R^2}\right]=\underbrace{\frac{Q}{2\pi\epsilon_0 R^2}}_{-E_0}x$$

The potential of the sphere of the field is

$$\varphi = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(R-x)^2+y^2}} - \frac{1}{\sqrt{(R+x)^2+y^2}} - \frac{a}{R}\frac{1}{\sqrt{(x-R')^2+y^2}} + \frac{a}{R}\frac{1}{\sqrt{(x+R')^2+y^2}}\right] =$$

$R' = \frac{a^2}{R} \ll 1$

$$\frac{1}{R\sqrt{1+\frac{r^2}{R^2}+2\frac{r}{R}\cos\theta}}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{R^2+r^2-2Rr\cos\theta}} - \frac{1}{\sqrt{R^2+r^2+2Rr\cos\theta}}\right.$$

$$\frac{1}{r\sqrt{1+\frac{R'^2}{r^2}+2\frac{R'}{r}\cos\theta}}$$

$$\left. - \frac{a}{R}\left(\frac{1}{\sqrt{R'^2+r^2-2R'r\cos\theta}} - \frac{1}{\sqrt{R'^2+r^2+2R'r\cos\theta}}\right)\right] \simeq$$

$$\simeq \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R}\left(1+\frac{r}{R}\cos\theta\right) - \frac{1}{R}\left(1-\frac{r}{R}\cos\theta\right) - \frac{a}{Rr}\frac{2R'}{r}\cos\theta\right] =$$

(2.10) $$= \frac{Q}{2\pi\varepsilon_0 R^2}\left[r\cos\theta - \frac{a^3}{r^2}\cos\theta\right] = -E_0\Big(\underbrace{r\cos\theta}_{\text{uniform field}} - \underbrace{\frac{a^3}{r^2}\cos\theta}_{\text{dipole field}}\Big)$$

Consider potential of a dipole:

$$\varphi_d = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-x_0)^2+y^2}} - \frac{1}{\sqrt{(x+x_0)^2+y^2}}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r\sqrt{1+\frac{x_0^2}{r^2}-\frac{2x_0}{r}\cos\theta}} - \frac{1}{r\sqrt{1+\frac{x_0^2}{r^2}+\frac{2x_0}{r}\cos\theta}}\right]$$

(2.11) $$\simeq \frac{Q}{4\pi\varepsilon_0 r^2}\left(-2x_0\cos\theta\right) = -\frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

p

$\Rightarrow$ Induced dipole moment:

(2.12) $$\frac{\vec{p}}{4\pi\varepsilon_0} = -\vec{E}_0 a^3 \Rightarrow \vec{p} = 4\pi\varepsilon_0 \vec{E}_0 a^3$$

Sphere @ arbitrary potential

Starting with the Green's function for pt. charge near the sphere,

$$(2.13) \quad G(\bar{r},\bar{r}_0) = \underbrace{\frac{1}{|\bar{r}-\bar{r}_0|}}_{\text{contribution due to}} - \frac{a}{r_0}\frac{1}{\left|\bar{r}-\frac{a^2}{r_0^2}\bar{r}_0\right|}$$

In spherical coordinates:

$$G(r,r_0) = \frac{1}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}} - \frac{a}{r_0}\frac{1}{\sqrt{r^2+a^4/r_0^2-2a^2\frac{r}{r_0}\cos\theta}}$$

$$= \frac{1}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}} - \frac{1}{\sqrt{a^2+\frac{r^2r_0^2}{a^2}-2rr_0\cos\theta}}$$

Note that $\begin{cases} G(r,r_0) = G(r_0,r) \\ G(r,a) = G(a,r_0) = 0 \end{cases}$

inside sphere

$$(2.14) \quad \frac{\partial G}{\partial n_0} = \pm\frac{\partial G}{\partial r_0}\bigg|_{r_0=a} = \mp\frac{a-r\cos\theta}{\sqrt{r^2+a^2-2ra\cos\theta}} \pm \frac{\frac{r^2}{a}-r\cos\theta}{\cdots}$$

outside sphere

$$= \pm\frac{r^2-a^2}{a\sqrt{r^2+a^2-2ra\cos\theta}}$$

$\Rightarrow$ potential due to charge distribution & sphere

$$(2.15) \quad \boxed{\Phi(r) = \int d^3r_0\, G(r,r_0)\rho(r_0) \pm \frac{1}{4\pi}\oint \Phi(r_0)\frac{\partial G}{\partial n_0}\underbrace{d^2a}_{a^2\cdot d\Omega_0}}$$

Separation of variables

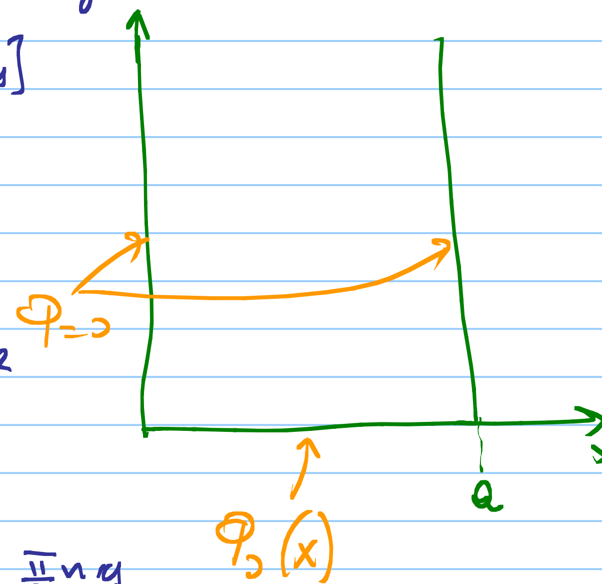
In systems with well-defined symmetry, separation of variables works well:

→ Consider a 2D rectangular well

(2.16) $$\begin{cases} \Delta\varphi = 0 \quad [\text{No free charges}] \\ \varphi = X(x)\,Y(y) \end{cases}$$

$$\Delta\varphi = X''Y + Y''X = 0$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -m^2$$



$$X = \sin \underbrace{\frac{n\pi}{a}}_{m_n} x$$

$$Y'' = m^2 Y \Rightarrow Y_m = e^{\pm\frac{\pi}{a} n y}$$

Assuming finite solution:

(2.17) $$\varphi = \sum_n \varphi_n \sin \frac{n\pi}{a} x \, e^{-\frac{\pi}{a} n y}$$

To find φ_n, use min square fit or Fourier expansion

$$\Delta = \int_0^a \left[\sum_n \varphi_n \sin \frac{n\pi}{a} x - \varphi_0(x)\right]^2 dx \to \min$$

$$\frac{\partial\Delta}{\partial\varphi_\ell} = \int_0^a \frac{\partial}{\partial\varphi_\ell}\left[\sum_n \varphi_n \sin \frac{n\pi}{a} x - \varphi_0(x)\right]^2 dx = 0$$

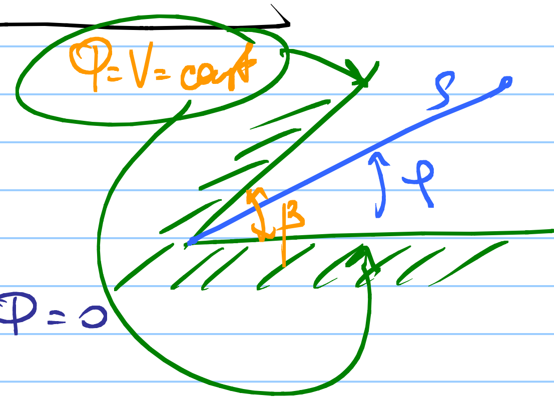
$$\int_0^a \left[\sum_n \varphi_n \sin \frac{n\pi}{a} x - \varphi_0(x)\right] \sin \frac{\pi\ell}{a} x \, dx = 0$$

$$\sum_n \varphi_n \underbrace{\int_0^a \sin \frac{n\pi}{a} x \sin \frac{\ell\pi}{a} x \, dx}_{A_{n\ell}} = \int_0^a \varphi_0(x) \sin \frac{\pi\ell}{a} x \, dx$$

$A_{nl} = I \cdot \frac{a}{2}$; in general, A_{nl} may not be diagonal

(2.18) $$\varphi_l = \frac{2}{a}\int_0^a \Phi_0(x)\, \sin\frac{\pi l}{a} x \, dx$$

corner, angle β



(2.19) $$\begin{cases} \Phi = R(\rho)\cdot\Psi(\varphi) \\ \Delta\Phi = \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right]\Phi = 0 \end{cases}$$

$$\frac{\Psi}{\rho}\left[R' + \rho R''\right] + \frac{R}{\rho^2}\Psi'' = 0$$

$$\underbrace{\frac{\rho R' + \rho^2 R''}{R}}_{m^2} + \underbrace{\frac{\Psi''}{\Psi}}_{-m^2} = 0$$

(2.20) $$\begin{cases} m \neq 0: \quad \Psi_m = e^{\pm i m\varphi}; \quad R(\rho) = \rho^{\pm m} \\ m = 0 \quad \Psi_m = \alpha_0 + \alpha_1\varphi; \quad R(\rho) = \rho_0 + \rho_1 \ln\rho \end{cases}$$

Since $\Phi(0) \neq \infty$: $\Phi(\rho,\varphi) = \alpha_0 + \sum_m \alpha_m \sin(m\varphi + \gamma_m)\,\rho^m$

Chose $m = \frac{\pi n}{\beta}$ $\Rightarrow$

$$\Phi(\rho,\varphi) = V + \sum_n \alpha_n \sin\frac{\pi n}{\beta}\varphi\cdot\rho^{\frac{\pi n}{\beta}}$$

(2.21) $$\boxed{\Phi(\rho\simeq 0,\varphi) \simeq V + \alpha_1 \sin\frac{\pi}{\beta}\varphi\, \rho^{\pi/\beta}}$$

Note:

(2.22) $$\begin{cases} E_\rho = -\frac{\partial\Phi}{\partial\rho} \simeq -\alpha_1 \sin\frac{\pi}{\beta}\varphi\ \frac{\pi}{\beta}\rho^{\pi/\beta - 1} \\ E_\varphi = -\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi} \simeq -\alpha_1\frac{\pi}{\beta}\cos\frac{\pi}{\beta}\varphi\ \rho^{\pi/\beta - 1} \\ \sigma|_\Gamma = \sigma(\rho,\varphi=0) = \sigma(\rho,\varphi=\beta) = \epsilon_0 E_\varphi \simeq -\frac{\epsilon_0\pi}{\beta}\rho^{\pi/\beta - 1} \end{cases}$$