TOPOLOGICAL INVARIANCE OF THE COMBINATORIAL EULER CHARACTERISTIC OF O-MINIMAL SETS

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ABSTRACT. We prove the topological invariance of the combinatorial Euler characteristic of o-minimal sets with the help of a canonical, topologically defined stratification of o-minimal sets by locally compact ones.

Introduction. Let $\mathcal{B}_n$ be the collection of semi-algebraic subsets of $\mathbb{R}^n$, i.e. sets definable by a finite boolean combination of polynomial equalities and inequalities. The $\mathcal{B}_n$, $n \in \mathbb{N}$, satisfy (i) any element of $\mathcal{B}_1$ is a finite union of open intervals (possibly infinite) and singletons; (ii) if $X \in \mathcal{B}_n$ and $\mathbb{R}^n \xrightarrow{\text{pr}} \mathbb{R}^m$ is any projection, then $\text{pr}(X) \in \mathcal{B}_m$.

(i) is immediate while (ii) is the Tarski-Seidenberg theorem. An *o-minimal* (short for *order-minimal*) structure over $\mathbb{R}$ is a non-trivial family of boolean algebras of subsets of $\mathbb{R}^n$ satisfying (i) and (ii). (See van den Dries [vdD98] for the meaning of *non-trivial* and a minimal set of axioms.) Starting from the 90's, remarkable examples of o-minimal structures have been discovered, both over the reals and other linearly ordered groups. Over $\mathbb{R}$, one can intuitively think of these as the result of permitting special families of real-analytic functions besides polynomials to serve in the equations and inequalities defining subsets.

Given an o-minimal structure $\mathcal{S}$, one has the associated notion of o-minimal function (a function whose graph belongs to $\mathcal{S}$); the product and coproduct of o-minimal sets are o-minimal. Let $K(\mathcal{S})$ be the Grothendieck ring of the category of $\mathcal{S}$-minimal sets and functions. There is a homomorphism

$$\text{eu} : K(\mathcal{S}) \longrightarrow \mathbb{Z} \qquad (\clubsuit)$$

that we will call the *combinatorial Euler characteristic*. Whenever $\mathcal{S}$ contains all semi-algebraic sets, eu is in fact an isomorphism. When $\mathcal{S}$ is the collection of semi-linear sets, $K(\mathcal{S})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and eu is the homomorphism $\langle m, n \rangle \mapsto m + n$.

The map ($\clubsuit$) has a long history. It starts, of course, with the proof of triangulability of algebraic and analytic varieties by Hironaka and Łojasiewicz. Over real closed fields, triangulations of semi-algebraic sets by open affine cells were constructed by Knebusch and Delfs [DK82]. In the context of semi-linear sets, eu and the determination of the Grothendieck semiring are due to Schanuel [Sch91]. Independently, van den Dries [vdD98] found a remarkable construction of eu that works for all o-minimal structures and avoids the use of triangulations in favor of the more order-theoretic cylindrical cell decompositions. For semi-algebraic sets, these were introduced by Collins; see Basu-Pollack-Roy [BPR06] Ch. 5. For reference, let us recall their definition following van den Dries [vdD98] Ch. 3. Following the usual model-theoretic convention, the term *definable* will mean *belonging to an o-minimal structure* (assumed fixed in the background).

Let $\langle i_1, i_2, \dots, i_m \rangle$ be a sequence of 0's and 1's. The collection of definable (cylindrical) $\langle i_1, i_2, \dots, i_m \rangle$-cells is given by induction on m. A $\langle 0 \rangle$-cell is a singleton in $\mathbb{R}$; a $\langle 1 \rangle$-cell is an open (possibly unbounded) interval in $\mathbb{R}$. If $\langle i_1, i_2, \dots, i_{m-1} \rangle$-cells in $\mathbb{R}^{m-1}$ have already been specified, an $\langle i_1, i_2, \dots, i_{m-1}, 0 \rangle$-cell is the graph of a continuous, definable, $\mathbb{R}$-valued function on some $\langle i_1, i_2, \dots, i_{m-1} \rangle$-cell X. An $\langle i_1, i_2, \dots, i_{m-1}, 1 \rangle$-cell is the set of points

$$\{(x, y) \in \mathbb{R}^m \mid x \in X, f(x) < y < g(x)\}$$

where X is an $\langle i_1, i_2, \dots, i_{m-1} \rangle$-cell and f, g are continuous, definable $\mathbb{R}$-valued functions on X with $f < g$. (Here $f \equiv -\infty$ or $g \equiv +\infty$ are also permitted.) The dimension of an $\langle i_1, i_2, \dots, i_m \rangle$-cell is $\sum_{k=1}^{m} i_k$. (For o-minimal structures over $\mathbb{R}$, this is the same as the topological dimension; in the axiomatic setting, this formula serves as a definition.)

Any definable X permits a decomposition into cylindrical cells and one can let

$$\mathrm{eu}(X) = \sum_{\alpha \in \mathsf{cell}} (-1)^{\dim(\alpha)}.$$

See van den Dries [vdD98] for why this gives a well-defined homomorphism $K(\mathcal{S}) \to \mathbb{Z}$.

Let $\mathcal{S}_1$, $\mathcal{S}_2$ be o-minimal structures over $\mathbb{R}$ and let A, B be an $\mathcal{S}_1$- resp. $\mathcal{S}_2$-definable set. The goal of this note is to prove

Theorem 1.1. *If A is homeomorphic to B, then* $\mathrm{eu}(A) = \mathrm{eu}(B)$.

O-minimal structures need not be unifiable. (See Rolin–Speissegger–Wilkie [RSW03] for the difficult proof.) In the theorem, it is not assumed that $\mathcal{S}_1$ and $\mathcal{S}_2$ have a common o-minimal extension. Of course, the generality afforded by stating the theorem in the above form is partly a mirage. If $\mathcal{S}_1$ and $\mathcal{S}_2$ both extend semi-algebraic sets, then there is an $\mathcal{S}_1$-definable homeomorphism between A and a semi-linear set (i.e. a triangulation of A), and an $\mathcal{S}_2$-definable triangulation of B. So in that case, the theorem is equivalent to its special case, the topological invariance of the combinatorial Euler characteristic of semi-algebraic sets:

Corollary 1.2. *If A, B are affine semi-algebraic sets that are homeomorphic, then* $\mathrm{eu}(A) = \mathrm{eu}(B)$.

Remark 1.3. Milnor's counterexample to the polyhedral *Hauptvermutung* implies that even two compact polyhedra can be homeomorphic without being o-minimally homeomorphic (in any o-minimal structure containing them). But of course, for compact semi-algebraic sets X, the homeomorphism (and in fact, homotopy) invariance of $\mathrm{eu}(X)$ follows from the fact that it equals the cohomological Euler characteristic defined via singular cohomology, or any equivalent cohomology theory. For locally compact semi-algebraic sets, the homeomorphism (and in fact, proper homotopy) invariance of the combinatorial Euler characteristic follows from its equaling the cohomological Euler characteristic defined via sheaf cohomology with compact support, or any equivalent cohomology theory. So the interest of the theorem is for o-minimal but not locally compact sets. No cohomology theory seems to be known such that the alternating sum of its betti numbers equals $\mathrm{eu}(X)$ for all semi-algebraic sets, nor whether $\mathrm{eu}(X)$ is a proper homotopy invariant in general.

A result closely related to Cor. 1.2 was proved by McCrory and Parusinski by entirely different methods, cf. Remark A.7. of [MP97]: *Let $h : X \to X$ be a homeomorphism (not necessarily semi-algebraic) of semi-algebraic sets. Let $\phi \in F(X)$ be such that $\phi' = \phi \circ h \in F(X)$. Let $Y \subset X$ be a compact semi-algebraic subset such that $Y = h^{-1}(Y)$ is also semi-algebraic. Then*

$$\int_Y \phi = \int_{Y'} \phi'.$$

(Here $F(X)$ is the ring of semi-algebraically constructible functions and integration is with respect to Euler characteristic as measure.) Taking $Y = X$ and F to be the characteristic function of a semi-algebraic subset of X, this means that the combinatorial Euler characteristic of an embedded semi-algebraic set is invariant with respect to homeomorphisms that extend to some compact semi-algebraic neighborhood. The proof by McCrory and Parusinski uses the possibility of expressing any semi-algebraic set as a topologically defined boolean combination of (possibly larger) closed semi-algebraic subsets of the ambient space. The argument in this paper stays inside the given set, with the help of a topologically defined stratification of o-minimal sets.

Proof of Theorem 1.1. Given any topological space X, let us, as it were, try to extract its locally compact 'core' by a 'greedy algorithm'. That is, set

$$z(X) := \big\{x \in X \mid x \text{ has a compact neighborhood in } X\big\}.$$

Lemma 1.4. *(i) $z(X)$ is open (possibly empty) in X. (ii) If X is Hausdorff, then $z(X)$ is locally compact.*

Proof. (i) If $x \in U \subseteq C$ with U open and C compact, then $U \subseteq z(X)$ as well. (ii) For all $y \in C - U$, let V_y, W_y be disjoint opens in X with $x \in V_y$, $y \in W_y$. Since $C - U$ is compact, there is a finite $I \subseteq C - U$ such that $\{W_i \mid i \in I\}$ covers $C - U$. Then

$$x \;\in\; U \cap \big(\bigcap_{i\in I} V_i\big) \;\subseteq\; C - \big(\bigcup_{i\in I} W_i\big) \;\subseteq\; U \;\subseteq\; z(X).$$

But $U \cap \big(\bigcap_{i\in I} V_i\big)$ is open and $C - \big(\bigcup_{i\in I} W_i\big)$ is compact. □

Remark 1.5. A general topological space need not have a maximal locally compact subspace, and $z(X)$ could well be empty even if X has locally compact subsets. (Consider, for example, the rationals with the metric topology.) Thankfully, $z(-)$ is well-behaved on o-minimal sets over $\mathbb{R}$.

By a stratification $\mathcal{S}$ of a topological space X let us mean merely a finite decomposition $X = \bigsqcup_{\alpha\in\mathcal{S}} \alpha$ such that the closure $\overline{\alpha}$ of any stratum is a union of strata. A stratum α is *maximal* if $\alpha \subseteq \overline{\beta}$ implies $\alpha = \beta$.

Lemma 1.6. *If X permits a stratification with locally compact strata, then $z(X)$ contains all maximal strata.*

Indeed, for a point x belonging to some stratum α, if $\alpha \not\subseteq \overline{\beta}$, then x cannot belong to the closure of β. Thus a maximal α is open in X and a compact neighborhood of x in α is a compact neighborhood of x in X.

Example 1.7. Consider $X = \alpha \sqcup \beta$ where α is the interior of the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ and β is the interval $[0,1]$. Then $z(X)$ is the union of α with $(0,1)$.

So it is not the case that $z(X)$ will be a union of strata in general, not even for a semi-algebraic X stratified (in the above, weak sense) into semi-algebraic manifolds. (This will not happen with stratifications extending to the closure of X in $\mathbb{R}^n$ though. See Prop. 1.12.)

By induction let $X_0 := z(X)$ and

$$X_i := z\big(X - (X_0 \cup X_1 \cup \cdots \cup X_{i-1})\big) \text{ for } i > 0. \tag{$*$}$$

Let us now work in some fixed (unnamed) o-minimal structure over $\mathbb{R}$. The following observation is enough to prove the theorem of the title.

Lemma 1.8. *Let X be definable. Then*

(i) *each X_i is definable*
(ii) $\dim\big(X - (X_0 \cup X_1 \cup \cdots \cup X_{i-1})\big) > \dim\big(X - (X_0 \cup X_1 \cup \cdots \cup X_i)\big)$ *as long as the space on the left is non-empty*
(iii) *the iteration terminates, providing a stratification $X = \sqcup_{i=1}^{N} X_i$ of X into finitely many locally compact definable subsets.*

Proof. (i) Thanks to locally compact and locally closed being the same for subsets of $\mathbb{R}^n$, $z(X)$ is first-order definable via

$$z(X) = \big\{x \in X \mid \text{ there is an } \epsilon > 0 \text{ such that for all } y \in \overline{B}(x,\epsilon) \\ \text{if } y \notin X \text{ then there is a } \delta > 0 \text{ such that } B(y,\delta) \cap X = \varnothing\big\}$$

where $B(x,r)$ is the open and $\overline{B}(x,r)$ the closed ball of radius r centered at x. (Replace 'ball' with 'box' if it is desirable to work over the structure $\langle \mathbb{R}, < \rangle$.) Of course, each stage of the iteration $(*)$ is also first-order definable.

(ii) Any definable set permits a stratification $\mathcal{S}$ into cylindrical cells. Since those are locally compact, this is precisely the situation of Lemma 1.6. Hence $z(X)$ contains all maximal cells. In particular, if $d = \dim(X)$, it contains all d-dimensional cells in $\mathcal{S}$. $X - z(X)$ is a definable subset of a union of cells of dimension less than d, hence

$$\dim(X) = \dim\big(z(X)\big) > \dim\big(X - z(X)\big)$$

Iterating this establishes the claim.

Remark 1.9. The best bound for the N in (iii) is actually $1 + \lfloor \frac{\dim(X)}{2} \rfloor$ rather than the obvious $\dim(X)$; see Cor. 1.14.

(iii) That the decomposition terminates follows from (i) and (ii); that the pieces are locally compact follows from Lemma 1.4. To show it is a stratification, note that $z(X)$ is open in X (cf. Lemma 1.4) but for X definable, $z(X)$ is also dense in X (since it contains all maximal cells). Iterating, X_i is relatively open and dense in $\big(X - (X_0 \cup X_1 \cup \cdots \cup X_{i-1})\big)$ which is closed in X. That is to say, $\overline{X_i} = \sqcup_{i \leqslant j \leqslant N} X_j$. □

We will call the output of the lemma the *intrinsic locally compact stratification* of X.

Proof of Theorem 1.1. Let $h : A \to B$ be a homeomorphism. Since the stages of the intrinsic locally compact stratification are defined purely topologically, h restricts to bijections (hence, homeomorphisms) $h_i : A_i \to B_i$. By the additivity of eu, it thus suffices to prove the theorem under the additional assumption that A, B are locally compact.

This should be well-known, but let us include the proof for completeness. Let K be any field and let $H_c^*(X,K)$ be sheaf cohomology with compact support with constant coefficients K. (All spaces will be assumed Hausdorff.) Let $\chi_c(X) := \sum_{i=0}^{\infty}(-1)^i \dim_K H_c^i(X,K)$ be the cohomological Euler characteristic with compact support. Let us again work in an arbitrary o-minimal structure over $\mathbb{R}$.

Proposition 1.10. *If X is definable and locally compact, then* $\mathrm{eu}(X) = \chi_c(X)$.

This is a consequence of three facts:

(1) For cylindrical cells C, $\mathrm{eu}(C) = (-1)^{\dim C}$ by definition and $\chi_c(C) = (-1)^{\dim C}$ since C is homeomorphic to $(0,1)^{\dim C}$.

(2) If $X = U \sqcup Z$ is any decomposition of a definable X into definable U, Z, then

$$\mathrm{eu}(X) = \mathrm{eu}(U) + \mathrm{eu}(Z).$$

(3) If $X = U \sqcup Z$ is an open-closed decomposition of a locally compact space X, and if the total cohomology $H_c^*(U,K)$ as well as $H_c^*(Z,K)$ are finite-dimensional, then the total cohomology $H_c^*(X,K)$ is finite-dimensional too; hence the corresponding Euler characteristics are well-defined, and in fact

$$\chi_c(X) = \chi_c(U) + \chi_c(Z).$$

(Cf. Iversen [Ive86] III.7.6.) For a locally compact definable X, one can then apply induction on a cylindrical cell decomposition of X, taking away one maximal (a fortiori, open) cell at a time. This finishes the proof of Thm. 1.1. ☐

Though the intrinsic locally compact stratification of X is not compatible with all stratifications of X into locally compact strata, it is compatible with (and coarsest) among those that are part of a stratification of an ambient locally compact space. This is a purely topological fact, but an immediate consequence is that for definable sets the intrinsic stratification goes 'twice as fast' as expected. (Cor. 1.14 below can also be proved directly using the geometry of cylindrical decompositions or triangulations.)

In what follows, lowercase greek letters will always denote strata; to unclutter notation, let us write $\alpha \preccurlyeq \beta$ for $\alpha \subseteq \overline{\beta}$ and $\alpha \prec \beta$ for $\alpha \subseteq \overline{\beta} - \beta$.

Lemma 1.11. *Let W be an arbitrary topologically stratified space and Z a union of strata.*

(i) *Z is open in W if and only if for every $\alpha \subseteq Z$, $\beta \subseteq W$, if $\alpha \prec \beta$ then $\beta \subseteq Z$.*
(ii) *Z is closed in W if and only if for every $\beta \subseteq W$, $\gamma \subseteq Z$, if $\beta \prec \gamma$ then $\beta \subseteq Z$.*
(iii) *Z is locally closed in W if and only if for every $\alpha, \gamma \subseteq Z$, $\beta \subseteq W$, if $\alpha \prec \beta \prec \gamma$ then $\beta \subseteq Z$.*

Proof. (ii) is saying that Z is a union of closures of strata. (Note that stratifications are always assumed to be finite.) (i) says that Z is the complement of a union of closures of strata.

(iii), *if*: Let $Z^{\uparrow} = \{\beta \in W \mid \alpha \preccurlyeq \beta \text{ for some } \alpha \in Z\}$; let $Z_{\downarrow} = \{\beta \subseteq W \mid \beta \preccurlyeq \gamma \text{ for some } \gamma \subseteq Z\}$. $Z_{\downarrow}$ is closed by (ii), $Z^{\uparrow}$ is open by (i). $Z \subseteq Z^{\uparrow} \cap Z_{\downarrow}$ clearly. $Z \supseteq Z^{\uparrow} \cap Z_{\downarrow}$ by the given condition.

(iii), *only if*: Suppose $Z = U \cap V$ with U open and V closed, α, β, γ as assumed. $\beta \subseteq V$ since $\beta \prec \gamma$ and V is closed. $\beta \cap U \neq \varnothing$ since $\alpha \prec \beta$ and U is open. Thence $\beta \cap Z \neq \varnothing$. So $\beta \subseteq Z$ since Z is a union of strata. □

The next proposition says that as long as the stratification of X is part of the stratification of an ambient locally compact space, $z(X)$ is the union of the 'top intervals' in the poset of strata of X.

Proposition 1.12. *Let W be Hausdorff, locally compact and topologically stratified. Let X be a union of strata. Define*

$$\mathsf{top}(X) = \text{union of } \{\alpha \subseteq X \mid \text{ for all } \beta \subseteq W \text{ and } \gamma \subseteq X, \text{ if } \alpha \prec \beta \prec \gamma \text{ then } \beta \subseteq X\}.$$

Then $z(X) = \mathsf{top}(X)$.

Remark 1.13. The condition defining $\mathsf{top}(X)$ may be satisfied vacuously too. For example, if α is maximal in X then $\alpha \subseteq \mathsf{top}(X)$.

Proof. $z(X) \supseteq \mathsf{top}(X)$: Lemma 1.11(i) implies that $\mathsf{top}(X)$ is open in X. (Indeed, let $\alpha \subseteq \mathsf{top}(X)$, $\beta \subseteq X$ with $\alpha \prec \beta$ and let $\beta \prec \beta_1 \prec \beta_2$ with $\beta_1 \subseteq W$, $\beta_2 \subseteq X$. $\beta_1 \subseteq X$ since $\alpha \subseteq \mathsf{top}(X)$. But that means $\beta \subseteq \mathsf{top}(X)$ by the definition of $\mathsf{top}(X)$.)

Lemma 1.11(iii) implies that $\mathsf{top}(X)$ is locally closed in W. (Indeed, let $\alpha, \gamma \subseteq \mathsf{top}(X)$, $\beta \subseteq W$, $\alpha \prec \beta \prec \gamma$. $\beta \subseteq X$ since $\alpha \subseteq \mathsf{top}(X)$ and then $\beta \subseteq \mathsf{top}(X)$ as before.)

Write $\mathsf{top}(X) = U \cap V$ with U open, V closed in W. Given $x \in \mathsf{top}(X)$, find a compact neighborhood K of x in W with $K \subseteq U$. Then $K \cap V$ is a compact neighborhood of x in $\mathsf{top}(X)$, a fortiori a compact neighborhood of x in X.

$z(X) \subseteq \mathsf{top}(X)$: say $x \in z(X)$ and $x \in \alpha \subseteq X$. Let $\alpha \prec \beta \prec \gamma$ be arbitrary with $\beta \subseteq W$, $\gamma \subseteq X$. $z(X)$ is locally compact by Lemma 1.4, hence locally closed in W. Write $z(X) = U \cap V$ with U open and V closed in W. Let γ_1 be maximal in X and $\gamma \preccurlyeq \gamma_1$; then $\gamma_1 \subseteq \mathsf{top}(X) \subseteq z(X) \subseteq V$. So $\beta \subseteq V$. $\beta \cap U \neq \varnothing$ since U is a neighborhood of x. So $\beta \cap z(X) \neq \varnothing$. Since X is a union of strata, $\beta \subseteq X$. But this means $\alpha \subseteq \mathsf{top}(X)$. □

Corollary 1.14. *Let X be definable. Then as long as the space on the left is non-empty,*

$$\dim(X_i) - \dim(X_{i+1}) \geqslant 2.$$

(Let us take $\dim(\varnothing) = -\infty$ by convention.)

Proof. If $X \subseteq \mathbb{R}^n$, find a stratification of $\mathbb{R}^n$ into cylindrical cells that partitions X (cf. van den Dries [vdD98] Ch. 4 Prop. 1.13). If $d = \dim(X)$ then $\mathsf{top}(X)$ contains all cells of dimension d and of $d-1$ too. Apply Prop. 1.12 and iterate. □

Cor. 1.14 cannot be improved further:

Example 1.15. Let P_i, $i = 0, 1, 2, \ldots, 2n$, be points in general position in $\mathbb{R}^{2n}$, let $\mathsf{int}\langle P_0, P_1, \ldots, P_{2i}\rangle$ be the relative interior of the simplex $\langle P_0, P_1, \ldots, P_{2i}\rangle$ and let

$$X = \bigsqcup_{i=0}^{n} \mathsf{int}\langle P_0, P_1, \ldots, P_{2i}\rangle.$$

Apply Prop. 1.12 with W the simplex $\langle P_0, P_1, \ldots, P_{2n}\rangle$ stratified into the relative interiors of all its subsimplices. That shows $X_i = \mathsf{int}\langle P_0, P_1, \ldots, P_{2(n-i)}\rangle$, of dimension $2(n-i)$ in turn.

For general X, of course, the strata will not be equidimensional. In fact, it could happen that for all $0 \leqslant i \leqslant N$ and all $0 \leqslant d \leqslant \dim(X_i)$, the stratum X_i contains points where the local dimension is d.

Cohomological ruminations. The fact that the intrinsic locally compact stratification is topological might tempt one to introduce, as a "poor man's substitute" of a cohomology theory realizing the combinatorial Euler characteristic

$$H^*_{\mathsf{str}}(X, K) := \bigoplus_{i=0}^{N} H^*_c(X_i, K).$$

Then H^*_{str} is homeomorphism-invariant and $\mathrm{eu}(X) = \chi_{\mathsf{str}}(X)$ for all definable X. H^*_{str} is obviously contravariantly functorial with respect to stratification-preserving maps that are proper on each stratum. It is also invariant with respect to such homotopies. Such maps are rare. I do not know an intrinsic (i.e. non-iterative) condition on a map $Y \xrightarrow{f} X$ that is equivalent to its respecting the intrinsic locally compact stratification. And of course, even the semi-algebraic inclusion of a compact subspace (which is then necessarily a proper map) can cut across all strata. Just consider the space X of Ex. 1.15. Let Q_i be the barycenter of the simplex $\langle P_0, P_1, \ldots, P_{2i}\rangle$ and let Y be the convex hull of $Q_0, Q_1, \ldots, Q_n$. Then $Y = Y_0$ and $Y_0 \cap X_i$ is non-empty for each i.

For X not locally compact, the textbook theory of cohomology with compact support does not provide a ready comparison between $H^*_{\mathsf{str}}(X)$ and $H^*_c(X)$, nor an easy way to prove or disprove whether $\mathrm{eu}(X) = \chi_c(X)$. The reason is, ultimately, that the classical repertoire of homological algebra — Mayer-Vietoris sequences, tautness, Künneth formula, extension by zero and so on — works best for sheaf cohomology with support in a paracompactifying family, and the family of compact subsets of X is paracompactifying if and only if X is (Hausdorff and) locally compact. Sheaf cohomology with a non-paracompactifying family of supports can indeed be paradoxical; for example, the cohomological dimension of $\mathbb{R}^n$ is $n+1$ if one allows all families of supports. (See Bredon [Bre97].) Perhaps the cohomological formalism that is best adopted to all definable sets will be one similar to perverse sheaves.

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