Differential Equations - Spring 2024

Exam 3

Wednesday, April 17, 10:00 am - 10:50 am

Your name (please print): ______________________________________

Instructions: This is a closed book, closed notes exam. The use of any electronic devices including calculators is not permitted. The exam consists of **5** problems and this booklet contains **7** pages (including this one). **Problems 1 through 3 are short answer questions and no partial credit will be given. On problems 4 through 5, you must show your work and justify your assertions to receive full credit**. Justify your answers and simplify your results as much as possible. Also, please clearly mark your final (simplified) answer. The last two pages of this booklet are blank. Good Luck!

FERPA Waiver: By my signature I relinquish my FERPA rights in the following context: My exam may be returned en masse with others present in the classroom. I acknowledge that I understand my score may be visible to others. If I choose not to relinquish my FERPA rights, I understand that I will have to present my student ID at my instructor's office to retrieve my exam.

Signature:______________________________________

No work required (No partial credit)

1. Set up the appropriate form of a particular solution y_p, but do not determine the value of coefficients

   (a) (**3 points**) $y'' + 4y = 2x + 1$

   $$y_c = c_1 \cos 2x + c_2 \sin 2x$$

   $$y_p(x) = Ax + B$$

   (b) (**3 points**) $u'' + u' = 2e^{3x}$

   $$u_c = c_1 + c_2 e^{-x}$$
   $$u_p = Ae^{3x}$$

   (c) (**3 points**) $x'' + 4x = 2\cos(2t)$

   $$x_c = c_1 \cos 2t + c_2 \sin 2t$$

   $$x_p = A\cos 2t + B\sin 2t (\text{ same as } y_c)$$

   Modify the guess by multiplying t

   $$x_p = t(A\cos 2t + B\sin 2t)$$

   (d) (**3 points**) $y'' + 4y' + 4y = e^{-3t} + 2t^2 + 1$

   $$y_c = c_1 e^{-2t} + c_2 t e^{-2t}$$

   $$y_p = Ae^{-3t} + Bt^2 + Ct + D$$

   (e) (**3 points**) $y'' + 4y' + 3y = xe^{-2x} + \cos(2x)$

   $$y_c = c_1 e^{-x} + c_2 e^{-3x}$$

   $$y_p = (Ax + B)e^{-2x} + C\cos(2x) + D\sin(2x)$$

Show your work

2. Find the general solution of (x' denotes derivative of $x(t)$ with respect to t)

$$x'' - 4x = 2e^{2t}$$

(a) (**3 points**) Find the complement solution y_c by solving

$$x'' - 4x = 0$$

$$r^2 - 4 = 0 \to r = \pm 2$$

$$x_c(t) = c_1 e^{-2t} + c_2 e^{2t}$$

(b) (**5 points**) Find a particular solution using the method of undetermined coefficients

$$x'' - 4x = 2e^{2t}$$

Let $x_p = Ae^{2t}$ (same as x_c). Correct the guess by multiplying t as

$$x_p = Ate^{2t}$$

$$x_p' = Ae^{2t} + 2Ate^{2t}$$

$$x_p'' = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t} = 4Ae^{2t} + 4Ate^{2t}$$

Substitute x_p, x_p', and x_p'' into the differential equation

$$x'' - 4x = 2e^{2t}$$

$$\to (4Ae^{2t} + 4Ate^{2t}) - 4(Ate^{2t}) = 2e^{2t}$$

$$\to 4Ae^{2t} = 2e^{2t} \to A = \frac{1}{2}$$

$$x_p(t) = \frac{1}{2}te^{2t}$$

(c) (**2 points**) Write the general solution

$$x(t) = c_1 e^{-2t} + c_2 e^{2t} + \frac{1}{2}te^{2t}$$

Show your work

3. (**10 points**) A mass $m = 2$ is attached to a spring with spring constant $k = 50$ and damping coefficient $c = 12$. The mass is set in motion with initial position $u(0) = 1$ and initial velocity $u'(0) = -7$. Find $u(t)$ in the form of $Ce^{-\alpha t}\cos(\omega_0 t - \delta)$ and identify the time-varying amplitude and pseudo-period.

$$m = 2, c = 12, k = 50, u(0) = 0, u'(0) = -7$$

$$2u'' + 12u' + 50u = 0$$

$$r^2 + 6r + 25 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 4(1)(25)}}{2} = \frac{-6 \pm \sqrt{-64}}{2} = -3 \pm 4i$$

General solution (underdamped)

$$u(t) = Ae^{-3t}\cos(4t) + Be^{-3t}\sin(4t)$$

$$u(0) = Ae^{-0}\cos 0 + Be^{0}\sin 0 = 0 \to A = 1$$

$$u'(0) = -3Ae^{0}\cos 0 - 4Ae^{0}\sin 0 - 3Be^{0}\sin 0 + 4Be^{0}\cos 0$$

$$= -3Ae^{0}\cos 0 + 4Be^{0}\cos 0 = -3A + 4B = -7 \to -3 + 4B = -7 \to B = -1$$

Thus,

$$u(t) = e^{-3t}\cos 4t - e^{-3t}\sin 4t$$

let

$$C = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Then

$$u(t) = \sqrt{2}e^{-3t}(\frac{1}{2}\cos(4t) + \frac{-1}{2}\sin(4t)) = \sqrt{2}e^{-3t}(\cos(4t)\cos\delta + \sin(4t)\sin\delta) = \sqrt{2}e^{-3t}\cos(4t - \delta)$$

where

$$\tan\delta = -1 \to \delta = -\frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

Finally,

$$u(t) = \sqrt{2}e^{-3t}\cos(4t + \frac{\pi}{4}) \text{ or } u(t) = \sqrt{2}e^{-3t}\cos(4t - \frac{3\pi}{4})$$

Time-varying amplitude $= \sqrt{2}e^{-3t}$

Pesudo-period $= \frac{2\pi}{4}$

Extra page for scratch work. I will not grade work on this page unless you write on another page"problem continued on page 5".

Extra page for scratch work. I will not grade work on this page unless you write on another page "problem continued on page 6".