Differential Equations
Homework 9
Due April 10, 2024, 9:59 am

Note:

- Please show all of your work (writing a list of answers is not sufficient).
- Please indicate the people you worked with.
- **Please staple your HW.**
- Several random problems will be graded (1 point each).

1. Find a general solution of

   (a)
$$y'' + 16y = e^{3x}$$
$$y = y_c + c_p$$

   1. y_c
$$y'' + 16y = 0 \to r^2 + 16 = 0 \to r = \pm 4i$$
$$y_c = C_1 \cos(4x) + C_2 \sin(4x)$$

   2. y_p
   Let
$$y_p = Ae^{3x}$$
   Then,
$$y_p' = 3Ae^{3x}, y_p'' = 9Ae^{3x},$$
   and
$$y'' + 16y = e^{3x} \to 9Ae^{3x} + 16Ae^{3x} = e^{3x} \to 25Ae^{3x} = e^{3x} \to A = \frac{1}{25}$$
   Thus
$$y_p = \frac{1}{25}e^{3x}$$
   and the general solution is
$$y = C_1 \cos(4x) + C_2 \sin(4x) + \frac{1}{25}e^{3x}$$

(b)

$$y'' - y' - 2y = 3x + 4$$

$$y = y_c + c_p$$

1. y_c

$$y'' - y' - 2y = 0 \to r^2 - r - 2 = 0 \to (r-2)(r+1) = 0 \to r = -1, 2$$

$$y_c = C_1 e^{-x} + C_2 e^{2x}$$

2. y_p

Let

$$y_p = Ax + B$$

Then,

$$y_p' = A, y_p'' = 0,$$

and

$$y'' - y' - 2y = 3x + 4 \to 0 - A - 2(Ax + B) = 3x + 4 \to \underbrace{-2A}_{=3} x \underbrace{-(A + 2B)}_{=4} = 3x + 4$$

$$-2A = 3, \quad -(A + 2B) = 4 \to A = -\frac{3}{2}, B = -\frac{5}{4}$$

Thus

$$y_p = -\frac{3}{2}x - \frac{5}{4} = -\frac{1}{4}(6x + 5)$$

and the general solution is

$$y = C_1 e^{-x} + C_2 e^{2x} - \frac{1}{4}(6x + 5)$$

(c)

$$y'' - y' - 6y = 2\sin(3x)$$

$$y = y_c + c_p$$

1. y_c

$$y'' - y' - 6y = 0 \to r^2 - r - 6 = 0 \to (r-3)(r+2) = 0 \to r = -2, 3$$

$$y_c = C_1 e^{-2x} + C_2 e^{3x}$$

2. y_p

Let

$$y_p = A\cos(3x) + B\sin(3x)$$

Then,

$$y_p' = -3A\sin(3x) + 3B\cos(3x), y_p'' = -9A\cos(3x) - 9B\sin(3x),$$

and

$$y'' - y' - 6y = 2\sin(3x)$$

$$\to (-9A\cos(3x) - 9B\sin(3x)) - (-3A\sin(3x) + 3B\cos(3x)) - 6(A\cos(3x) + B\sin(3x)) = 2\sin(3x)$$

$$\to \underbrace{(-15A - 3B)}_{=0}\cos(3x) + \underbrace{(-15B + 3A)}_{=2}\sin(3x) = 2\sin(3x)$$

$$\to -15A - 3B = 0, 3A - 15B = 2$$

$$A = \frac{1}{39}, B = -\frac{5}{39}$$

Thus

$$y_p = \frac{1}{39}\cos(3x) - \frac{5}{39}\sin(3x)$$

and the general solution is

$$y = C_1 e^{-2x} + C_2 e^{3x} + \frac{1}{39}\cos(3x) - \frac{5}{39}\sin(3x)$$

(d)

$$y'' + 2y' - 3y = 1 + xe^x$$

$$y = y_c + c_p$$

1. y_c

$$y'' + 2y' - 3y = 0 \to r^2 + 2r - 3 = 0 \to (r+3)(r-1) = 0 \to r = 1, -3$$

$$y_c = C_1e^x + C_2e^{-3x}$$

2. y_p

Let

$$y_p = A + (Bx + C)e^x = A + Bxe^x + \underbrace{Ce^x}_{\text{part of } y_c} \quad \text{(not working)}$$

Let

$$y_p = A + x(Bx + C)e^x = A + (Bx^2 + Cx)e^x = A + Bx^2e^x + Cxe^x \text{ (working)}$$

Then,

$$y_p' = (2Bx + C)e^x + (Bx^2 + Cx)e^x = Bx^2e^x + (2B + C)xe^x + Ce^x$$

$$y_p'' = 2Be^x + (2Bx+C)e^x + (2Bx+C)e^x + (Bx^2+Cx)e^x = (2B+2C)e^x + (4B+C)xe^x + Bx^2e^x$$

and

$$y'' + 2y' - 3y = 1 + xe^x$$

$$\to ((2B+2C)e^x + (4B+C)xe^x + Bx^2e^x) + 2(Bx^2e^x + (2B+C)xe^x + Ce^x) - 3(A + Bx^2e^x + Cxe^x)$$

$$= 1 + xe^x$$

$$\to \underbrace{-3A}_{=1} + \underbrace{(2B + 4C)}_{=0}e^x + \underbrace{(8B)}_{=1}xe^x = 1 + xe^x$$

$$\to -3A = 1, \quad (2B + 4C) = 0, \quad 8B = 1$$

$$A = -\frac{1}{3}, B = \frac{1}{8}, C = -\frac{1}{16}$$

Thus

$$y_p = -\frac{1}{3} + \frac{1}{8}x^2e^x - \frac{1}{16}xe^x$$

and the general solution is

$$y = C_1e^x + C_2e^{-3x} - \frac{1}{3} + \frac{1}{8}x^2e^x - \frac{1}{16}xe^x$$

(e)

$$y'' + 9y = 2\cos(3x) + 3\sin(3x)$$

$$y = y_c + c_p$$

1. y_c

$$y'' + 9y = 0 \to r^2 + 9 = 0 \to r = \pm 3i$$

$$y_c = C_1 \cos(3x) + C_2 \sin(3x)$$

2. y_p

Let

$$y_p = A\cos(3x) + B\sin(3x)(\text{same as } y_c) \to \text{ not working}$$

Let

$$y_p = x(A\cos(3x) + B\sin(3x)).$$

Then

$$y_p' = (A\cos(3x) + B\sin(3x)) + 3x(-A\sin(3x) + B\cos(3x))$$

$$y_p'' = (-3A\sin(3x)+3B\cos(3x))+3(-A\sin(3x)+B\cos(3x))+9x(-A\cos(3x)-B\sin(3x))$$

and

$$y'' + 9y = 2\cos(3x) + 3\sin(3x)$$

$$\to (-3A\sin(3x)+3B\cos(3x))+3(-A\sin(3x)+B\cos(3x))+9x(-A\cos(3x)-B\sin(3x))+9x(A\cos(3x)+B\sin(3x))$$

$$= 2\cos(3x) + 3\sin(3x)$$

$$\to \underbrace{6B}_{=2}\cos(3x) \underbrace{-6A}_{=3}\sin(3x) = 2\cos(3x) + 3\sin(3x)$$

$$\to 6B = 2, -6A = 3$$

$$B = \frac{1}{3}, A = -\frac{1}{2}$$

Thus

$$y_p = -\frac{1}{2}x\cos(3x) + \frac{1}{3}x\sin(3x)$$

and the general solution is

$$y = C_1\cos(3x) + C_2\sin(3x) - \frac{1}{2}x\cos(3x) + \frac{1}{3}x\sin(3x)$$

2. Set up the appropriate form of a particular solution y_p, but do not determine the value of coefficients

(a)

$$y'' - 2y' + 2y = e^x \sin(x)$$

1. y_c

$$y'' - 2y' + 2y = 0 \to r^2 - 2r + 2 = 0 \to r = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm i$$

$$y_c = C_1 e^x \cos x + C_2 e^x \sin x$$

2. y_p

$$y_p = Ae^x(B\cos x + C\sin x) = De^x \cos x + Ee^x \sin x$$

It is same as y_c. Thus, multiply x and obtain

$$y_p = xe^x(D\cos x + E\sin x)$$

(b)

$$y^{(4)} + 5y'' + 4y = \sin(x) + \cos(2x)$$

1. y_c

$$y^{(4)} + 5y'' + 4y = 0 \to r^4 + 5r^2 + 4 = 0 \to (r^2+1)(r^2+4) = 0 \to r = \pm i, \pm 2i$$

$$y_c = C_1 \cos x + C_2 \sin x + C_3 \cos(2x) + C_4 \sin(2x)$$

2. y_p

$$y_p = A\sin x + B\cos x + C\cos(2x) + D\sin(2x)$$

It is same as y_c. Thus, multiply x and obtain

$$y_p = x(A\sin x + B\cos x + C\cos(2x) + D\sin(2x))$$

3. Solve the initial value problem

(a)

$$y'' + 4y = 2x, \quad y(0) = 1, y'(0) = 2$$

$$y = y_c + y_p$$

1. y_c

$$y'' + 4y = 0 \to r^2 + 4 = 0 \to r = \pm 2i$$

$$y_c = C_1 \cos(2x) + C_2 \sin(2x)$$

1. y_p

Let

$$y_p = Ax + B$$

Then,

$$y_p' = A, y_p'' = 0$$

and

$$y'' + 4y = 2x \to 0 + 4(Ax + B) = 2x \to \underbrace{4A}_{=2} x + \underbrace{4B}_{=0} = 2x$$

$$4A = 2, 4B = 0$$

$$\to A = \frac{1}{2}, B = 0$$

$$y_p = \frac{1}{2}x$$

The general solution is

$$y = y_c + y_p = C_1 \cos(2x) + C_2 \sin(2x) + \frac{1}{2}x$$

Using the initial condition

$$y(0) = C_1 \cos 0 + C_2 \sin 0 + \frac{1}{2}0 = 1 \to C_1 = 1$$

$$y'(0) = -2C_1 \sin 0 + 2C_2 \cos 0 + \frac{1}{2} = 2 \to C_2 = \frac{3}{4}$$

Therefore,

$$y = \cos(2x) + \frac{3}{4}\sin(2x) + \frac{1}{2}x$$

(b)

$$y'' + 9y = \sin(2x), \quad y(0) = 1, y'(0) = 0$$

$$y = y_c + y_p$$

1. y_c

$$y'' + 9y = 0 \to r^2 + 9 = 0 \to r = \pm 3i$$

$$y_c = C_1 \cos(3x) + C_2 \sin(3x)$$

1. y_p

Let

$$y_p = A\cos(2x) + B\sin(2x)$$

Then,

$$y_p' = -2A\sin(2x) + 2B\cos(2x), y_p'' = -4A\cos(2x) - 4B\sin(2x)$$

and

$$y'' + 9y = \sin(2x)$$

$$\to -4A\cos(2x) - 4B\sin(2x) + 9(A\cos(2x) + B\sin(2x)) = \sin(2x)$$

$$\to \underbrace{5A}_{=0}\cos(2x) + \underbrace{5B}_{=1}\sin(2x) = \sin(2x) \to A = 0, B = \frac{1}{5}$$

$$y_p = \frac{1}{5}\sin(2x)$$

The general solution is

$$y = y_c + y_p = C_1\cos(3x) + C_2\sin(3x) + \frac{1}{5}\sin(2x).$$

Using the initial condition

$$y(0) = C_1\cos 0 + C_2\sin 0 + \frac{1}{5}\sin 0 = 1 \to C_1 = 1$$

$$y'(0) = -3C_1\sin 0 + 3C_2\cos 0 + \frac{2}{5}\cos 0 = 3C_2 + \frac{2}{5} = 0 \to C_2 = -\frac{2}{15}$$

Therefore,

$$y = \cos(3x) - \frac{2}{15}\sin(3x) + \frac{1}{5}\sin(2x)$$