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Differential Equations
Homework 11 (Optional)
Due 4/25 (Thurs) 11:59am

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Note:

- Please show all of your work (writing a list of answers is not sufficient).

- Please indicate the people you worked with.

- **Please staple your HW.**

- Several random problems will be graded (1 point each).

1. Use the <u>definition</u> to find the Laplace transform of

   (a)
   $$f(t) = t$$

   By integration by parts

   $$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} t\, dt = \lim_{A \to \infty} \int_0^A e^{-st} t\, dt = \lim_{A \to \infty} [-\frac{1}{s} e^{-st} t]_0^A + \lim_{A \to \infty} \frac{1}{s} \int_0^A e^{-st} dt$$

   $$= \lim_{A \to \infty} \underbrace{(-\frac{1}{s} e^{-sA} A}_{\to 0 \text{ if } s>0} + \underbrace{\frac{1}{s} e^{-s0} 0)}_{=0} + \lim_{A \to \infty} \frac{1}{s} \int_0^A e^{-st} dt = \lim_{A \to \infty} \frac{1}{s} \int_0^A e^{-st} dt$$

   $$= \lim_{A \to \infty} [-\frac{1}{s^2} e^{-st}]_0^A = \lim_{A \to \infty} \underbrace{(-\frac{1}{s^2} e^{-sA}}_{\to 0} + \frac{1}{s^2} e^{-s0}) = \frac{1}{s^2}, s > 0$$

   (b)
   $$f(t) = e^{3t+1}$$

   $$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} e^{3t+1} dt = e(\lim_{A \to \infty} \int_0^A e^{(3-s)t} dt)$$

   $$= \frac{e}{3-s} \lim_{A \to \infty} [e^{(3-s)t}]_0^A = \frac{e}{3-s} \lim_{A \to \infty} [\underbrace{e^{(3-s)A}}_{\to 0 \text{ if } 3-s<0} - \underbrace{e^{(3-s)0}}_{=1}] = \frac{e}{3-s}(-1) = \frac{e}{s-3}, s > 3$$

   (c)
   $$f(t) = \begin{cases} 1, & 0 < t \le 1 \\ 0, & t > 1 \end{cases}$$

   $$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \int_0^1 e^{-st}(1) dt + \int_1^\infty e^{-st}(0) dt$$

   $$= [-\frac{1}{s} e^{-st}]_0^1 = -\frac{e^{-s}}{s} + \frac{1}{s} = \frac{1 - e^{-s}}{s}, s \ne 0$$

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1

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2. Use the <u>table</u> to find the Laplace transform of

(a)
$$f(t) = t - 2e^{3t}$$
$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - 2\mathcal{L}\{e^{3t}\} = \frac{1}{s^2} - \frac{2}{s-3}, s > 3$$

(b)
$$f(t) = \sin(2t) + \cos(2t)$$
$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 2t\} + \mathcal{L}\{\cos 2t\} = \frac{2}{s^2+4} + \frac{s}{s^2+4} = \frac{s+2}{s^2+4}, s > 0$$

(c)
$$f(t) = \cos^2 2t$$

(Hint: Use double angle formula)

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\cos^2 2t\} = \mathcal{L}\{\frac{1+\cos 4t}{2}\} = \mathcal{L}\{\frac{1}{2}\} + \frac{1}{2}\mathcal{L}\{\cos 4t\}$$

$$= \frac{1}{2s} + \frac{1}{2}\frac{s}{s^2+16}, s > 0$$

3. Use the <u>table</u> to find the inverse Laplace transform of

(a)
$$F(s) = \frac{3s+1}{s^2+4}$$
$$f(t) = \mathcal{L}^{-1}\{\frac{3s+1}{s^2+4}\} = 3\mathcal{L}^{-1}\{\frac{s}{s^2+4}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{s^2+4}\}$$
$$= 3\cos(2t) + \frac{1}{2}\sin(2t)$$

(b)
$$F(s) = \frac{5-3s}{s^2+9}$$
$$f(t) = \mathcal{L}^{-1}\{\frac{5-3s}{s^2+9}\} = 5\mathcal{L}^{-1}\{\frac{1}{s^2+3^2}\} - 3\mathcal{L}^{-1}\{\frac{s}{s^2+3^2}\} = \frac{5}{3}\mathcal{L}^{-1}\{\frac{3}{s^2+3^2}\} - 3\mathcal{L}^{-1}\{\frac{s}{s^2+3^2}\}$$
$$= \frac{5}{3}\sin(3t) - 3\cos(3t)$$

(c)
$$F(s) = 2s^{-1}e^{-3s}$$
$$f(t) = \mathcal{L}^{-1}\{2s^{-1}e^{-3s}\} = 2\mathcal{L}^{-1}\{\frac{1}{s}e^{-3s}\} = 2u_3(t)$$

2

4. Use Laplace transforms to solve the initial value problem

(a)
$$x'' + 4x = 0; x(0) = 5, x'(0) = 0$$

By taking Laplace transform

$$(s^2 X - sx(0) - x'(0)) + 4X = 0$$

$$(s^2 X - 5s) + 4X = 0 \rightarrow (s^2 + 4)X = 5s$$

$$X = \frac{5s}{s^2 + 4}$$

$$x(t) = \mathcal{L}^{-1}\{\frac{5s}{s^2 + 4}\} = 5\mathcal{L}^{-1}\{\frac{s}{s^2 + 4}\} = 5\cos(2t)$$

(b)
$$x'' - x' - 2x = 0, ; x(0) = 0, x'(0) = 2$$

$$(s^2 X - sx(0) - x'(0)) - (sX - x(0)) - 2X = 0$$

$$(s^2 X - 2) - (sX) - 2X = 0$$

$$(s^2 - s - 2)X = 2$$

$$X = \frac{2}{(s^2 - s - 2)} = \frac{2}{(s - 2)(s + 1)} = \frac{2/3}{s - 2} - \frac{2/3}{s + 1} \text{ (using partial fraction)}$$

$$x(t) = \mathcal{L}^{-1}\{\frac{2/3}{s - 2} - \frac{2/3}{s + 1}\} = \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t}$$

(c)
$$x'' + x = \cos(3t); x(0) = 1, x'(0) = 0$$

$$(s^2 X - sx(0) - x'(0)) + X = \frac{s}{s^2 + 9}$$

$$(s^2 X - s) + X = \frac{s}{s^2 + 9}$$

$$(s^2 + 1)X = \frac{s}{s^2 + 9} + s$$

$$X = \frac{s}{(s^2 + 9)(s^2 + 1)} + \frac{s}{s^2 + 1}$$

Partial fraction

$$\frac{s}{(s^2 + 9)(s^2 + 1)} = \frac{As + B}{s^2 + 9} + \frac{Cs + D}{s^2 + 1} = \frac{(As + B)(s^2 + 1) + (Cs + D)(s^2 + 9)}{(s^2 + 9)(s^2 + 1)}$$

$$= \frac{As^3 + As + Bs^2 + B + Cs^3 + 9Cs + Ds^2 + 9D}{(s^2 + 9)(s^2 + 1)}$$

3

$$= \frac{(A+C)s^3 + (B+D)s^2 + (A+9C)s + (B+9D)}{(s^2+9)(s^2+1)}$$

$$A + C = 0, B + D = 0, A + 9C = 1, B + 9D = 0$$

$$A = -C \rightarrow A + 9C = 1 \rightarrow -C + 9C = 1 \rightarrow C = \frac{1}{8}, A = -\frac{1}{8}$$

$$B = -D \rightarrow B + 9D = 0 \rightarrow -D + 9D = 0 \rightarrow D = 0, B = 0$$

Therefore,

$$X = \frac{s}{(s^2+9)(s^2+1)} + \frac{s}{s^2+1} = -\frac{1}{8}\frac{s}{s^2+9} + \frac{1}{8}\frac{s}{s^2+1} + \frac{s}{s^2+1}$$

$$x(t) = \mathcal{L}^{-1}\{X\} = \mathcal{L}^{-1}\{-\frac{1}{8}\frac{s}{s^2+3^2} + \frac{1}{8}\frac{s}{s^2+1^2} + \frac{s}{s^2+1^2}\} = -\frac{1}{8}\cos(3t) + \frac{1}{8}\cos t + \cos t$$

$$= \frac{9}{8}\cos t - \frac{1}{8}\cos(3t)$$

(d)

$$x'' + 3x' + 2x = t; x(0) = 0, x'(0) = 2$$

$$(s^2X - sx(0) - x'(0)) + 3(sX - x(0)) + 2X = \frac{1}{s^2}$$

$$(s^2X - 2) + 3sX + 2X = \frac{1}{s^2}$$

$$X(s^2 + 3s + 2) = \frac{1}{s^2} + 2$$

$$X = \frac{1}{s^2(s^2+3s+2)} + \frac{2}{s^2+3s+2}$$

$$X = \frac{1}{s^2(s+2)(s+1)} + \frac{2}{(s+2)(s+1)} = \frac{-\frac{3}{4}s + \frac{1}{2}}{s^2} + \frac{-\frac{1}{4}}{s+2} + \frac{1}{s+1} + \frac{-2}{s+2} + \frac{2}{s+1}$$

$$= -\frac{3}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2} + \frac{1}{s+1} - \frac{2}{s+2} + \frac{2}{s+1}$$

$$x(t) = -\frac{3}{4} + \frac{1}{2}t - \frac{1}{4}e^{-2t} + e^{-t} - 2e^{-2t} + 2e^{-t}$$

$$= -\frac{3}{4} + \frac{1}{2}t - \frac{9}{4}e^{-2t} + 3e^{-t} = \frac{1}{4}(-3 + 2t - 9e^{-2t} + 12e^{-t})$$

4