Differential Equations
Homework 10 (Revised)
(HW 10 will Not be collected but one of the problems will be in Midterm 3)
(HW 10 solution will be available on 4/15)

Note:

- Please show all of your work (writing a list of answers is not sufficient).
- Please indicate the people you worked with.
- **Please staple your HW.**
- Several random problems will be graded (1 point each).

1. Determine the period and frequency of the simple harmonic motion (damping coefficient $c = 0$) of a $m = 0.75$ kg mass on the end of a spring with spring constant $k = 48$.

$$m = 0.75, k = 48$$

$$mu'' + ku = 0$$

$$0.75u'' + 48u = 0$$

$$u'' + 64u = 0$$

$$r^2 + 64 = 0 \to r = \pm 8i$$

General solution

$$u(t) = A\cos(8t) + B\sin(8t) = C\cos(8t - \delta),$$

where

$$C = \sqrt{A^2 + B^2}, \tan\delta = \frac{B}{A}$$

The period

$$T = \frac{2\pi}{8} = \frac{\pi}{4}$$

The frequency

$$f = \frac{1}{T} = \frac{4}{\pi}$$

2. A mass of $m = 3$ kg is attached to the end of a spring that is stretched 0.2 m by a force of 15 N. At a time $t = 0$ the body is pulled 1 m, stretching the spring, and set in motion with an initial velocity of -10 m/s.

   (a) Find $u(t)$

$$F = -kx \to -15 = -k(0.2) \to k = \frac{15}{0.2} = 75$$

$$u(0) = 1, u'(0) = -10$$

$$m = 3, k = 75$$

$$3u'' + 75u = 0$$

$$u'' + 25u = 0$$

$$r^2 + 25 = 0 \to r = \pm 5i$$

   General solution

$$u(t) = A\cos(5t) + B\sin(5t)$$

$$u(0) = A\cos 0 + B\sin 0 = 0 \to A = 1$$

$$u'(0) = -5A\sin(0) + 5B\cos(0) = -10 \to B = -2$$

   Therefore

$$u(t) = \cos(5t) - 2\sin(5t) = \sqrt{1^2 + (-2)^2}\cos(5t - \delta) = \sqrt{5}\cos(5t - \delta)$$

   where

$$\tan\delta = -\frac{2}{1}$$

   (b) Find the amplitude and period of motion of the body.
   Amplitude $= \sqrt{5}$, Period $= \frac{2\pi}{5}$, frequency $= \frac{5}{2\pi}$

3. Suppose that the mass in a mass-spring system with $m = 25$, $c = 10$, and $k = 226$ is set in motion with $u(0) = 20$ and $u'(0) = 41$.

   (a) Find the position function $u(t)$ in the form of single cosine function.

$$m = 25, c = 10, k = 226, u(0) = 20, u'(0) = 41$$

$$25u'' + 10u' + 226u = 0$$

$$25r^2 + 10r + 226 = 0$$

$$r = \frac{-10 \pm \sqrt{100 - 4(25)(226)}}{50} = \frac{-10 \pm \sqrt{-22500}}{50} = \frac{-10 \pm 150i}{50} = -\frac{1}{5} \pm 3i$$

General solution

$$u(t) = Ae^{-\frac{1}{5}t}\cos 3t + Be^{-\frac{1}{5}t}\sin 3t$$

$$u(0) = Ae^{-0}\cos 0 + Be^{0}\sin 0 = 20 \to A = 20$$

$$u'(0) = -\frac{1}{5}Ae^{0}\cos 0 - 3Ae^{0}\sin 0 - \frac{1}{5}Be^{0}\sin 0 + 3Be^{0}\cos 0$$

$$= -\frac{1}{5}Ae^{0}\cos 0 + 3Be^{0}\cos 0 = -4 + 3B = 41 \to B = 15$$

Thus,

$$u(t) = 20e^{-\frac{1}{5}t}\cos 3t + 15e^{-\frac{1}{5}t}\sin 3t$$

let

$$C = \sqrt{20^2 + (15)^2} = 25$$

Then

$$u(t) = 25e^{-\frac{1}{5}t}(\frac{20}{25}\cos 3t + \frac{15}{25}\sin 3t) = 25e^{-\frac{1}{5}t}(\cos 3t\cos\delta + \sin 3t\sin\delta) = 25e^{-\frac{1}{5}t}\cos(3t - \delta),$$

where

$$\tan\delta = \frac{15}{20} = \frac{3}{4}$$

$$\delta = \tan^{-1}(\frac{3}{4}) \approx 0.6435$$

Finally,

$$u(t) = 25e^{-\frac{1}{5}t}\cos(3t - 0.6435)$$

   (b) Find the pseudo-period of the oscillation and time-varying amplitude.

   pseudoperiod = $\frac{2\pi}{3}$, Time-varying amplitude= $25e^{-\frac{1}{5}t}$

4. A mass $m = 1/2$ is attached to a spring with spring constant $k = 4$ and damping coefficient $c = 3$. The mass is set in motion with initial position $u(0) = 2$ and initial velocity $u'(0) = 0$. Find u(t) and determine whether the motion is overdamped, critically damped, or underdamped. If it is underdapmed, write $u(t)$ in the form of $Ce^{-pt}\cos(\omega_0 t - \delta)$.

$$m = \frac{1}{2}, c = 3, k = 4, u(0) = 2, u'(0) = 0$$

$$\frac{1}{2}u'' + 3u' + 4u = 0$$

$$r^2 + 6r + 8 = 0$$

$$(r+2)(r+4) = 0 \rightarrow r = -2, r = -4$$

General solution (Overdamped)

$$u(t) = Ae^{-2t} + Be^{-4t}$$

$$u(0) = Ae^0 + Be^0 = 2 \rightarrow A + B = 2$$

$$u'(0) = -2Ae^0 - 4Be^0 = 0 \rightarrow A + 2B = 0$$

$$B = -2, A = 4$$

Thus,

$$u(t) = 4e^{-2t} - 2e^{-4t}$$

5. A mass $m = 2$ is attached to a spring with spring constant $k = 50$ and damping coefficient $c = 12$. The mass is set in motion with initial position $u(0) = 1$ and initial velocity $u'(0) = -7$. Find u(t) and determine whether the motion is overdamped, critically damped, or underdamped. If it is underdapmed, write $u(t)$ in the form of $Ce^{-pt}\cos(\omega_0 t - \delta)$.

$$m = 2, c = 12, k = 50, u(0) = 1, u'(0) = -7$$

$$2u'' + 12u' + 50u = 0$$

$$r^2 + 6r + 25 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 4(1)(25)}}{2} = \frac{-6 \pm \sqrt{-64}}{2} = -3 \pm 4i$$

General solution (underdamped)

$$u(t) = Ae^{-3t}\cos(4t) + Be^{-3t}\sin(4t)$$

$$u(0) = Ae^{-0}\cos 0 + Be^{0}\sin 0 = 0 \to A = 1$$

$$u'(0) = -3Ae^{0}\cos 0 - 4Ae^{0}\sin 0 - 3Be^{0}\sin 0 + 4Be^{0}\cos 0$$

$$= -3Ae^{0}\cos 0 + 4Be^{0}\cos 0 = -3 + 4B = -7 \to B = -1$$

Thus,

$$u(t) = e^{-3t}\cos 4t - e^{-3t}\sin 4t$$

let

$$C = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Then

$$u(t) = \sqrt{2}e^{-3t}(\frac{1}{\sqrt{2}}\cos(4t) + \frac{-1}{\sqrt{2}}\sin(4t)) = \sqrt{2}e^{-3t}(\cos(4t)\cos\delta + \sin(4t)\sin\delta) = \sqrt{2}e^{-3t}\cos(4t - \delta)$$

where

$$\tan\delta = \frac{B}{A} = -1 \to \delta = \frac{3\pi}{4}$$

Finally,

$$u(t) = \sqrt{2}e^{-3t}\cos(4t - \frac{3\pi}{4})$$

Time-varying amplitude $= \sqrt{2}e^{-3t}$
Pseudo-period $= \frac{2\pi}{4}$