Differential Equations - Spring 2024

Exam 2

Wednesday, Mar. 27, 10:00 am - 10:50 am

Your name (please print): ____

Instructions: This is a closed book, closed notes exam. The use of any electronic devices including calculators is not permitted. The exam consists of **5** problems and this booklet contains **8** pages (including this one). **On problems 1 through 5, you must show your work and justify your assertions to receive full credit**. Justify your answers and simplify your results as much as possible. Also, please clearly mark your final (simplified) answer. The last two pages of this booklet are blank. Good Luck!

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Signature:____

No work required (No partial credit)

1. (**5 points**) The differential equation

$$y^{(5)} - 9y^{(4)} + 35y^{(3)} - 71y'' + 68y' - 24y = 0$$

has its characteristic equation (when the solution is assumed as $y = e^{rx}$)

$$r^5 - 9r^4 + 35r^3 - 71r^2 + 68r - 24 = 0$$

The roots are $r = 1,\ 1,\ 2 \pm 2i,\ 3.$ Write the general solution of the differential equation.

$$y(x) = c_1 e^{x} + c_2 x e^{x} + c_3 e^{2x}\cos(2x) + c_4 e^{2x}\sin(2x) + c_5 e^{3x}$$

2. Determine if given pairs of functions are linearly dependent or independence

   (a) (**3 points**) $f(x) = e^{2x}$ and $g(x) = xe^{2x}$

   Answer : Linearly independent

   (b) (**3 points**) $f(x) = x^3$ and $g(x) = 2x^3$

   Answer: Linearly dependent

   (c) (**3 points**) $f(x) = 2^x$ and $g(x) = 2^{x+2}$

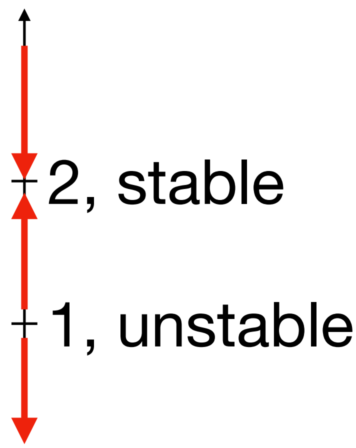
   Answer: Linearly dependent

No work required (No partial credit)

3. (**5 points**) Draw a phase diagram of

$$y' = \frac{1}{2}(y-2)(1-y)$$

and identify the equilibrium solutions and determine their stability (stable, unstable, or semistable).



Show your work

4. Find the general solution of

   (a) (**5 points**)

$$y'' - 5y' + 6y = 0$$

$$r^2 - 5r + 6 = 0 \rightarrow (r-3)(r-2) = 0 \rightarrow r = 2, 3$$

$$y = c_1e^{2x} + c_1e^{3x}$$

Answer: $y(x) =$____

(b) (**5 points**)

$$2y'' - 12y' + 18y = 0$$

$$2r^2 - 12r + 18 = 0 \to 2(r^2 - 6r + 9) = 0 \to 2(r-3)^2 = 0 \to r = 3 \text{ (repeated roots)}$$

$$y = c_1 e^{3x} + c_2 x e^{3x}$$

Answer: $y(x) =$____

(c) (**5 points**)

$$y'' - y' + y = 0$$

$$r^2 - r + 1 = 0 \to r = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$y = c_1 e^{\frac{1}{2}x} \cos \frac{\sqrt{3}}{2}x + c_2 e^{\frac{1}{2}x} \sin \frac{\sqrt{3}}{2}x$$

Answer: $y(x) =$____

Show your work

5. (**10 points**) Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to produce ($\beta(t) = 0$) and the death rate $\delta(t) = \frac{3}{\sqrt{P}}$. There were initially 900 fish ($P(0) = 400$) and the population of fish follows the general population model

$$\frac{dP}{dt} = (\beta(t) - \delta(t))P.$$

Find $P(t)$ and how long did it take all the fish in the lake to die?

$$\frac{dP}{dt} = -\frac{3}{\sqrt{P}}P = -3\sqrt{P} \to \frac{1}{\sqrt{P}}dP = -3dt \to \int \frac{1}{\sqrt{P}}dP = \int -3dt + C$$

$$2\sqrt{P} = -3t + C \to \sqrt{P} = -\frac{3t}{2} + C$$

$$\sqrt{900} = C \to C = 30$$

Thus

$$\sqrt{P} = -\frac{3t}{2} + 30 \to P = \left(-\frac{3t}{2} + 30\right)^2$$

$$0 = \left(-\frac{3t}{2} + 30\right)^2 \to -\frac{3t}{2} = -30 \to t = 20$$

Extra page for scratch work. I will not grade work on this page unless you write on another page"problem continued on page 6".

Extra page for scratch work. I will not grade work on this page unless you write on another page "problem continued on page 7".