Differential Equations
Homework 8 (Optional)
Due Mar. 25, 2024, 4:00 pm

Note:

- Please show all of your work (writing a list of answers is not sufficient).
- Please indicate the people you worked with.
- **Please staple your HW.**
- Several random problems will be graded (1 point each).

1. Find general solutions of the differential equations

   (a)
   $$y'' + 6y' + 9y = 0$$
   Let
   $$y = e^{rx}$$
   $$y' = re^{rx}$$
   $$y'' = r^2 e^{rx}$$
   Then, the differential equation becomes
   $$y'' + 6y' + 9y = 0 \to r^2 e^{rx} + 6re^{rx} + 9e^{rx} = 0 \to r^2 + 6r + 9 = 0$$
   $$(r+3)^2 = 0 \to r = -3(\text{repeated root}) \to y_1 = e^{-3x}, y_2 = xe^{-3x}$$
   General solution
   $$y = C_1 e^{-3x} + C_2 x e^{-3x}$$

(b)

$$4y'' - 12y' + 9y = 0$$

Let

$$y = e^{rx}$$
$$y' = re^{rx}$$
$$y'' = r^2 e^{rx}$$

Then, the differential equation becomes

$$4y'' - 12y' + 9y = 0 \to 4r^2 e^{rx} - 12re^{rx} + 9e^{rx} = 0 \to 4r^2 - 12r + 9 = 0$$

$$(2r-3)^2 = 0 \to r = \frac{3}{2}(\text{repeated root}) \to y_1 = e^{\frac{3}{2}x}, y_2 = xe^{\frac{3}{2}x}$$

General solution

$$y = C_1 e^{\frac{3}{2}x} + C_2 x e^{\frac{3}{2}x}$$

(c)

$$y'' - 6y' + 13y = 0$$

Let

$$y = e^{rx}$$
$$y' = re^{rx}$$
$$y'' = r^2e^{rx}$$

Then, the differential equation becomes

$$y'' - 6y' + 13y = 0 \to r^2e^{rx} - 6re^{rx} + 13e^{rx} = 0 \to r^2 - 6r + 13 = 0$$

$$r = \frac{6 \pm \sqrt{36 - 4(13)}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i \to y_1 = e^{3x}\cos 2x, y_2 = e^{3x}\sin 2x$$

General solution

$$y = C_1e^{3x}\cos 2x + C_2e^{3x}\sin 2x$$

(d)

$$y'' + 8y' + 25y = 0$$

Let

$$y = e^{rx}$$
$$y' = re^{rx}$$
$$y'' = r^2 e^{rx}$$

Then, the differential equation becomes

$$y'' + 8y' + 25y = 0 \to r^2 e^{rx} + 8re^{rx} + 25e^{rx} = 0 \to r^2 + 8r + 25 = 0$$

$$r = \frac{-8 \pm \sqrt{64 - 4(25)}}{2} = \frac{-8 \pm \sqrt{-36}}{2} = -4 \pm 3i \to y_1 = e^{-4x} \cos(3x), y_2 = e^{-4x} \sin(3x)$$

General solution

$$y = C_1 e^{-4x} \cos 3x + C_2 e^{-4x} \sin 3x$$

(e)

$$y^{(4)} + 3y'' - 4y = 0$$

Let

$$y = e^{rx}, y' = re^{rx}, y'' = r^2e^{rx}, y^{(3)} = r^3e^{rx}, y^{(4)} = r^4e^{rx}$$

Then, the differential equation becomes

$$y^{(4)} + 3y'' - 4y = 0 \to r^4e^{rx} + 3r^2e^{rx} - 4e^{rx} = 0 \to r^4 + 3r^2 - 4 = 0$$

$(r^2+4)(r^2-1) \to r = \pm 2i, r = -1, 1 \to y_1 = e^{0x}\cos 2x, y_2 = e^{0x}\sin 2x, y_3 = e^x, y_4 = e^{-x}$

General solution

$$y = C_1\cos 2x + C_2\sin 2x + C_3e^x + C_4e^{-x}$$

(f)

$$y^{(4)} - 8y'' + 16y = 0$$

Let

$$y = e^{rx}, y' = re^{rx}, y'' = r^2 e^{rx}, y^{(3)} = r^3 e^{rx}, y^{(4)} = r^4 e^{rx}$$

Then, the differential equation becomes

$$y^{(4)} - 8y'' + 16y = 0 \to r^4 e^{rx} - 8r^2 e^{rx} + 16e^{rx} = 0 \to r^4 - 8r^2 + 16 = 0$$

$$(r^2-4)^2 = 0 \to (r^2-4)(r^2-4) = 0, \to (r-2)(r+2)(r-2)(r+2) = 0 \to (r-2)^2(r+2)^2 = 0$$

$$\to r = -2 \text{ (repeated roots) }, 2 \text{ (repeated roots)}$$

$$\to y_1 = e^{-2x}, y_2 = xe^{-2x}, y_3 = e^{2x}, y_4 = xe^{2x}$$

General solution

$$y = C_1 e^{-2x} + C_2 x e^{-2x} + C_3 e^{2x} + C_4 x e^{2x}$$

2. Solve the initial value problems

(a)

$$9y'' + 6y' + 4y = 0, y(0) = 3, y'(0) = 4$$

Let

$$y = e^{rx}$$

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

Then, the differential equation becomes

$$9y' + 6y' + 4y = 0 \rightarrow 9r^2e^{rx} + 6re^{rx} + 4e^{rx} = 0 \rightarrow 9r^2 + 6r + 4 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 4(9)(4)}}{18} = \frac{-6 \pm \sqrt{-108}}{18} = -\frac{1}{3} \pm i\frac{1}{\sqrt{3}}$$

$$y_1 = e^{-\frac{x}{3}} \cos\left(\frac{x}{\sqrt{3}}\right), y_2 = e^{-\frac{x}{3}} \sin\left(\frac{x}{\sqrt{3}}\right)$$

General solution

$$y = c_1e^{-\frac{x}{3}} \cos\left(\frac{x}{\sqrt{3}}\right) + c_2e^{-\frac{x}{3}} \sin\left(\frac{x}{\sqrt{3}}\right)$$

$$y(0) = c_1e^{-\frac{0}{3}} \cos\left(\frac{0}{\sqrt{3}}\right) + c_2e^{-\frac{0}{3}} \sin\left(\frac{0}{\sqrt{3}}\right) = c_1 = 3$$

$$y = 3e^{-\frac{x}{3}} \cos\left(\frac{x}{\sqrt{3}}\right) + c_2e^{-\frac{x}{3}} \sin\left(\frac{x}{\sqrt{3}}\right)$$

$$y' = -e^{-\frac{x}{3}} \cos\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{\sqrt{3}}e^{-\frac{x}{3}} \sin\left(\frac{x}{\sqrt{3}}\right) - \frac{c_2}{3}e^{-\frac{x}{3}} \sin\left(\frac{x}{\sqrt{3}}\right) + \frac{c_2}{\sqrt{3}}e^{-\frac{x}{3}} \cos\left(\frac{x}{\sqrt{3}}\right)$$

$$y'(0) = -e^{-\frac{0}{3}} \cos\left(\frac{0}{\sqrt{3}}\right) - \frac{3}{\sqrt{3}}e^{-\frac{0}{3}} \sin\left(\frac{0}{\sqrt{3}}\right) - \frac{c_2}{3}e^{-\frac{0}{3}} \sin\left(\frac{0}{\sqrt{3}}\right) + \frac{c_2}{\sqrt{3}}e^{-\frac{0}{3}} \cos\left(\frac{0}{\sqrt{3}}\right) = -1 + \frac{c_2}{\sqrt{3}} = 4$$

$$c_2 = 5\sqrt{3}$$

Therefore

$$y = 3e^{-\frac{x}{3}} \cos\left(\frac{x}{\sqrt{3}}\right) + 5\sqrt{3}e^{-\frac{x}{3}} \sin\left(\frac{x}{\sqrt{3}}\right)$$

(b)

$$y^{(3)} + 10y'' + 25y' = 0, y(0) = 3, y'(0) = 4, y''(0) = 5$$

Let

$$y = e^{rx}, y' = re^{rx}, y'' = r^2e^{rx}, y^{(3)} = r^3e^{rx}$$

Then, the differential equation becomes

$$y^{(3)} + 10y'' + 25y' = 0 \to r^3e^{rx} + 10r^2e^{rx} + 25re^{rx} = 0 \to r^3 + 10r^2 + 25r = 0$$

$$r(r+5)^2 = 0 \to r = 0, -5, -5 \to y_1 = e^{0x}, y_2 = e^{-5x}, y_3 = xe^{-5x}$$

General solution

$$y = C_1 + C_2e^{-5x} + C_3xe^{-5x}$$

$$y(0) = C_1 + C_2 = 3$$

$$y'(0) = -5C_2 + C_3 = 4$$

$$y''(0) = 25C_2 - 10C_3 = 5$$

$$C_1 = \frac{24}{5}, C_2 = -\frac{9}{5}, C_3 = -5$$

$$y = \frac{24}{5} - \frac{9}{5}e^{-5x} - 5xe^{-5x}$$