Differential Equations
Homework 7
Due Mar. 20, 2024, 9:59 am

Note:

- Please show all of your work (writing a list of answers is not sufficient).
- Please indicate the people you worked with.
- **Please staple your HW.**
- Several random problems will be graded (1 point each).

1. Determine whether the pairs of functions are linearly independent or linearly dependent on the real line

$$f(x) = \pi, \quad g(x) = \cos^2 x + \sin^2 x$$

$$f(x) = \pi, g(x) = \cos^2 x + \sin^2 x = 1$$

$$g(x) = \frac{1}{\pi} f(x)$$

Linearly dependent

2. Determine whether the pairs of functions are linearly independent or linearly dependent on the real line

$$f(x) = 1 + x, \quad g(x) = 1 + |x|$$

If $x \geq 0 |x| = x$, then $f(x) = 1 + x, g(x) = 1 + x$, they are linearly dependent

If $x < 0 |x| = -x$, then $f(x) = 1 + x, g(x) = 1 - x$, they are linearly independent

On real line, they are linearly independent

3. Determine whether the pairs of functions are linearly independent or linearly dependent on the real line

$$f(x) = \sin^2 x, \quad g(x) = 1 - \cos 2x$$

$$f(x) = \sin^2 x, g(x) = 1 - \cos 2x = 2\sin^2 x$$

They are linearly dependent

4. Find general solutions of the differential equations

(a)

$$y'' - 3y' + 2y = 0$$

Let

$$y = e^{rx}$$

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

Then, the differential equation becomes

$$y'' - 3y' + 2y = 0 \to r^2 e^{rx} - 3re^{rx} + 2e^{rx} = 0 \to r^2 - 3r + 2 = 0$$

$$(r-2)(r-1) = 0 \to r = 1, 2 \to y_1 = e^x, y_2 = e^{2x}$$

General solution

$$y = C_1 e^x + C_2 e^{2x}$$

(b)

$$y'' + 5y' = 0$$

Let

$$y = e^{rx}$$

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

Then, the differential equation becomes

$$y' + 5y' = 0 \rightarrow r^2 e^{rx} + 5re^{rx} = 0 \rightarrow r^2 + 5r = 0$$

$$r(r+5) = 0 \rightarrow r = 0, -5 \rightarrow y_1 = e^{0x} = 1, y_2 = e^{-5x}$$

General solution

$$y = C_1 + C_2 e^{-5x}$$

(c)

$$2y'' - y' - y = 0$$

Let

$$y = e^{rx}$$

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

Then, the differential equation becomes

$$2y'' - y' - y = 0 \rightarrow 2r^2 e^{rx} - re^{rx} - e^{rx} = 0 \rightarrow 2r^2 - r - 1 = 0$$

$$(2r+1)(r-1) = 0 \rightarrow r = 1, -\frac{1}{2} \rightarrow y_1 = e^x, y_2 = e^{-\frac{1}{2}x}$$

General solution

$$y = C_1 e^x + C_2 e^{-\frac{1}{2}x}$$

5. Find a differential equation with a given general solution

(a)

$$y = c_1 e^{10x} + c_2 e^{-10x}$$
$$\rightarrow r = 10, -10$$

10 and -10 are roots of

$$(r - 10)(r + 10) = 0$$
$$r^2 - 100 = 0$$

r^2 comes from y''
1 comes from y

Therefore, differential equation should be

$$y'' - 100y = 0$$

(b)

$$y = c_1 e^{10x} + c_2 e^{100x}$$

$$\rightarrow r = 10, 100$$

10 and 100 are roots of

$$(r - 10)(r - 100) = 0$$

$$r^2 - 110r + 1000 = 0$$

r^2 comes from y''
r comes from y'
1 comes from y

Therefore, differential equation should be

$$y'' - 110y' + 1000y = 0$$