Differential Equations
Homework 6
Due Mar. 13, 2024, 9:59 am

Note:

- Please show all of your work (writing a list of answers is not sufficient).
- Please indicate the people you worked with.
- **Please staple your HW.**
- Several random problems will be graded (1 point each).

1. Solve the initial value problem using Euler's method with $h = 0.1$

$$y' = 3 + x - y, \quad y(0) = 1, \quad 0 \leq x \leq 0.5$$

$$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4,\ x_5 = 0.5$$

$$\begin{aligned}
&y(0) = 1\\
&y(0.1) \approx 1 + 0.1(3 + 0 - 1) = 1 + 0.2 = 1.2\\
&y(0.2) \approx 1.2 + 0.1(3 + 0.1 - 1.2) = 1.39\\
&y(0.3) \approx 1.39 + 0.1(3 + 0.2 - 1.39) = 1.571\\
&y(0.4) \approx 1.571 + 0.1(3 + 0.3 - 1.571) = 1.7439\\
&y(0.5) \approx 1.7439 + 0.1(3 + 0.4 - 1.7439) = 1.9095
\end{aligned}$$

2. Solve the initial value problem using Euler's method with $h = 0.1$

$$y' = 2y - 1, \quad y(0) = 1, \quad 0 \leq x \leq 0.5$$

$$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4,\ x_5 = 0.5$$

$$\begin{aligned}
&y(0) = 1 \\
&y(0.1) \approx 1 + 0.1(2 \cdot 1 - 1) = 1.1 \\
&y(0.2) \approx 1.1 + 0.1(2 \cdot 1.1 - 1) = 1.22 \\
&y(0.3) \approx 1.22 + 0.1(2 \cdot 1.22 - 1) = 1.3640 \\
&y(0.4) \approx 1.3640 + 0.1(2 \cdot 1.3640 - 1) = 1.5368 \\
&y(0.5) \approx 1.5368 + 0.1(2 \cdot 1.5368 - 1) = 1.7442
\end{aligned}$$

3. Verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = C_1 y_1 + C_2 y_2$ that satisfies the given initial conditions.

$$y'' - y = 0, \quad y_1 = e^x, \quad y_2 = e^{-x}, \quad y(0) = 0, \quad y'(0) = 5$$

$$y_1 = e^x, \; y_1' = e^x, \; y_1'' = e^x \to y_1'' - y_1 = e^x - e^x = 0$$
$$y_2 = e^{-x}, \; y_2' = -e^{-x}, \; y_1'' = e^{-x} \to y_2'' - y_2 = e^{-x} - e^{-x} = 0$$

Let $y(x) = C_1 e^x + C_2 e^{-x}$, then $y'(x) = C_1 e^x - C_2 e^{-x}$. From the initial conditions

$$y(0) = C_1 + C_2 = 0 \to C_2 = -C1$$
$$y'(0) = C_1 - C_2 = 5 \to C_1 + C_1 = 5 \to C_1 = \frac{5}{2}, C_2 = -\frac{5}{2}$$
$$y = \frac{5}{2}e^x - \frac{5}{2}e^{-x}$$

4. Verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the given initial conditions.

$$y'' + 6y' + 13y = 0, \quad y_1 = e^{-3x}\cos 2x, \quad y_2 = e^{-3x}\sin 2x, \quad y(0) = 2, \quad y'(0) = 0$$

$$\begin{aligned}
y_1 &= e^{-3x}\cos 2x \\
y_1' &= -3e^{-3x}\cos 2x - 2e^{-3x}\sin 2x \\
y_1'' &= 9e^{-3x}\cos 2x + 6e^{-3x}\sin 2x + 6e^{-3x}\sin 2x - 4e^{-3x}\cos 2x = 5e^{-3x}\cos 2x + 12e^{-3x}\sin 2x \\
&y_1'' + 6y_1' + 13y_1 \\
&\quad = (5e^{-3x}\cos 2x + 12e^{-3x}\sin 2x) + 6(-3e^{-3x}\cos 2x - 2e^{-3x}\sin 2x) + 13(e^{-3x}\cos 2x) = 0
\end{aligned}$$

$$\begin{aligned}
y_2 &= e^{-3x}\sin 2x \\
y_2' &= -3e^{-3x}\sin 2x + 2e^{-3x}\cos 2x \\
y_2'' &= 9e^{-3x}\sin 2x - 6e^{-3x}\cos 2x - 6e^{-3x}\cos 2x - 4e^{-3x}\sin 2x = -12e^{-3x}\cos 2x + 5e^{-3x}\sin 2x \\
&y_2'' + 6y_2' + 13y_2 \\
&\quad = (-12e^{-3x}\cos 2x + 5e^{-3x}\sin 2x) + 6(-3e^{-3x}\sin 2x + 2e^{-3x}\cos 2x) + 13(e^{-3x}\sin 2x) = 0
\end{aligned}$$

Let $y(x) = C_1 e^{-3x}\cos 2x + C_2 e^{-3x}\sin 2x$.

Then $y'(x) = (-3C_1 + 2C_2)e^{-3x}\cos(2x) - (2C_1 + 3C_2)e^{-3x}\sin(2x)$. From the initial conditions

$$\begin{aligned}
&y(0) = C_1 = 2 \\
&y'(0) = -6 + 2C_2 = 0 \rightarrow C_2 = 3 \\
&y = 2e^{-3x}\cos 2x + 3e^{-3x}\sin 2x
\end{aligned}$$

5. Verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions.

$$x^2y'' - xy' + y = 0, \quad y_1 = x, \quad y_2 = x\ln x, \quad y(1) = 7, \quad y'(1) = 2$$

$$y_1 = x,\ y_1' = 1,\ y_1'' = 0 \to x^2y_1'' - xy_1' + y_1 = 0 - x + x = 0$$
$$y_2 = x\ln x,\ y_2' = \ln x + 1,\ y_1'' = \frac{1}{x} \to x^2y_2'' - xy_2' + y_2 = x^2\frac{1}{x} - x(\ln x + 1) + x\ln x = 0$$

Let $y(x) = C_1x + C_2x\ln x$. Then $y'(x) = C_1 + (C_2 + 1)\ln x$. From the initial conditions

$$y(1) = C_1 = 7$$
$$y'(1) = 7 + C_2 = 2 \to C_2 = -5$$
$$y = 7x - 5x\ln x$$