Differential Equations - Spring 2024

Exam 1

Wednesday, Feb. 14, 10:00 am - 10:50 am

Your name (please print): ____

Instructions: This is a closed book, closed notes exam. The use of any electronic devices including calculators is not permitted. The exam consists of **5** problems and this booklet contains **?** pages (including this one). **On problems 1 through 5, you must show your work and justify your assertions to receive full credit**. Justify your answers and simplify your results as much as possible. Also, please clearly mark your final (simplified) answer. The last two pages of this booklet are blank. Good Luck!

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Signature:____

Show your work

1. (**10 points**) Find a function $y = f(x)$ satisfying

$$\frac{dy}{dx} = \frac{x}{y}, y(0) = -2$$

$$ydy = xdx$$

$$\int ydy = \int xdx + C$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y^2 = x^2 + C$$

$$y = \pm\sqrt{x^2 + C}$$

From $y(0) = -2$

$$y(0) = \pm\sqrt{C} = -2$$

$$+\sqrt{C} = -2 \text{ impossible}$$

$$-\sqrt{C} = -2 \rightarrow C = 4$$

Therefore

$$y = -\sqrt{x^2 + 4}$$

2. (**10 points**) Find the general solution of

$$y' + \frac{1}{x}y = 2e^{x^2+1}$$

$$y = \frac{1}{\mu}\int \mu 2e^{x^2+1}dx + \frac{C}{\mu},$$

where

$$\mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Thus

$$y = \frac{1}{x}\int x2e^{x^2+1}dx + \frac{C}{x},$$

Let $u = x^2 + 1$ and integrate

$$\begin{aligned} y &= \frac{1}{x}\int x2e^u \frac{1}{2x}du + \frac{C}{x} = \frac{1}{x}\int e^u du + \frac{C}{x} \\ &= \frac{1}{x}e^u + \frac{C}{x} = \frac{1}{x}e^{x^2+1} + \frac{C}{x} \end{aligned}$$

Therefore

$$y = \frac{1}{x}e^{x^2+1} + \frac{C}{x}$$

3. (**10 points**) Find the general solution of

$$(4x-y)+(6y-x)\frac{dy}{dx}=0$$

$$M=(4x-y)\rightarrow\frac{\partial M}{\partial y}=-1$$

$$N=(6y-x)\rightarrow\frac{\partial N}{\partial x}=-1$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, it is an exact equation. Let

$$\frac{\partial F}{\partial x}=M=4x-y \quad (1)$$

$$\frac{\partial F}{\partial y}=N=6y-x \quad (2)$$

Then,

$$(4x-y)+(6y-x)\frac{dy}{dx}=0\rightarrow\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0\rightarrow\frac{\partial}{\partial x}F(x,y)=0$$

By integration

$$F(x,y)=C.$$

The F can be found from Eqs. (1) and (2). By integration Eq. (1) with respect to x

$$F=\int 4x-ydx=2x^2-xy+g(y).$$

Its partial derivative with respect to y is

$$\frac{\partial F}{\partial y}=-x+g'(y)$$

and it should be equal Eq. (2). Thus

$$-x+g'(y)=6y-x\rightarrow g'(y)=6y\rightarrow g(y)=3y^2+D$$

Therefore

$$F(x,y)=2x^2-xy+3y^2+D.$$

Finally, the solution of differential equation is

$$F(x,y)=C\rightarrow 2x^2-xy+3y^2+D=C\rightarrow 2x^2-xy+3y^2=C$$

4. (**10 points**) Find the general solution of

$$y' - \frac{6}{x}y = -9y^{4/3}$$

$$\frac{1}{y^{4/3}}y' - \frac{6}{x}\frac{y}{y^{4/3}} = -9$$

$$\frac{1}{y^{4/3}}y' - \frac{6}{x}y^{-1/3} = -9$$

Let

$$v = y^{-1/3}$$

Then

$$v' = (-\frac{1}{3})y^{-4/3}y' \to y' = -3v'y^{4/3}$$

The differential equation transforms to

$$\frac{1}{y^{4/3}}y' - \frac{6}{x}y^{-1/3} = -9 \to \frac{1}{y^{4/3}}(-3v'y^{4/3}) - \frac{6}{x}v = -9 \to -3v' - \frac{6}{x}v = -9$$

$$v' + \frac{2}{x}v = 3$$

By using the integrating factor method

$$v = \frac{1}{\mu}\int 3\mu dx + \frac{C}{\mu},$$

where

$$\mu = e^{\int \frac{2}{x}dx} = x^2$$

Thus

$$v = \frac{1}{x^2}\int 3x^2 dx + \frac{C}{x^2} = \frac{1}{x^2}x^3 + Cx^{-2} = x + Cx^{-2},$$

Since $v = y^{-1/3}$

$$y^{-1/3} = x + Cx^{-2} \to y = (x + Cx^{-2})^{-3}$$

5. (**10 points**) A tank contains 1 gallons of water with 2 lb salt dissolved. Then, pure water is poured into the tank at a rate of 2 gallons per minute, and mixture is allowed to leave at 2 gallons per minute. Find the amount of salt $x(t)$ at time t

$$x(0) = 2, r_i = 2, c_i = 0, r_o = 2, c_0 = \frac{x(t)}{1}$$

$$\frac{dx}{dt} = 2 \cdot 0 - 2 \cdot \frac{x(t)}{1} = -2x(t)$$

$$\frac{dx}{dt} = -2x$$

By separation

$$\frac{1}{x}dx = -2dt \to x(t) = Ce^{-2t}$$

Since $x(0) = 2$,

$$x(0) = Ce^0 = 2 \to C = 2$$

Therefore

$$x(t) = 2e^{-2t}$$

Extra page for scratch work. I will not grade work on this page unless you write on another page“problem continued on page 7”.

Extra page for scratch work. I will not grade work on this page unless you write on another page "problem continued on page 8".