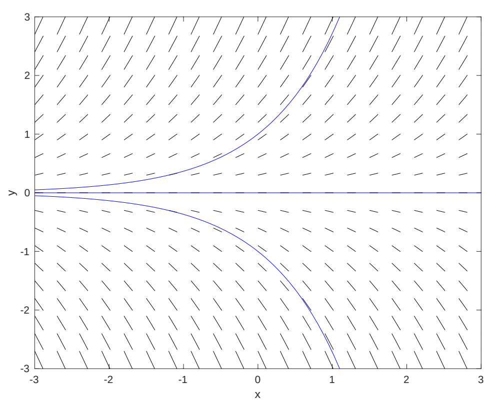
Differential Equations
Homework 5
Due Feb. 28, 2024, 9:59 am

Note:

- Please show all of your work (writing a list of answers is not sufficient).
- Please indicate the people you worked with.
- **Please staple your HW.**
- Several random problems will be graded (1 point each).

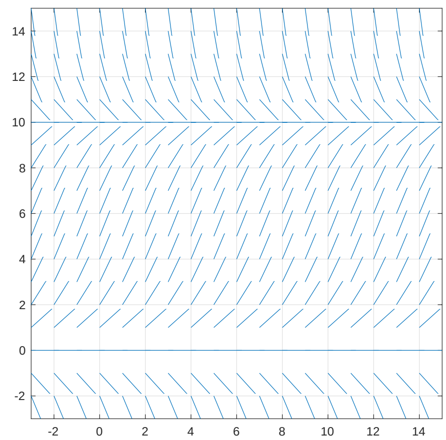
1. Sketch (by hands) slope field of

$$\frac{\partial y}{\partial t} = y$$



2. Sketch (by hands) slope field of

$$\frac{\partial p}{\partial t} = p(10 - p)$$



3. Separate variables and use partial fraction to solve the initial value problem

$$\frac{dx}{dt} = x - x^2, x(0) = 2$$

$$\frac{1}{x(1-x)}dx = dt$$

$$\int \frac{1}{x(1-x)}dx = \int 1dt + C$$

Partial fraction

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x} = \frac{A(1-x)+Bx}{x(1-x)} = \frac{A+(B-A)x}{x(1-x)}$$

$$A = 1, B - A = 0, \to A = 1, B = 1$$

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

Thus

$$\int \frac{1}{x(1-x)}dx = \int 1dt + C$$

becomes

$$\int (\frac{1}{x} + \frac{1}{1-x})dx = t + C$$

$$\ln(x) - \ln(1-x) = t + C \to \ln \frac{x}{1-x} = t + C$$

$$\frac{x}{1-x} = Ce^t$$

$$\frac{2}{1-2} = Ce^0 \to C = -2$$

$$\frac{x}{1-x} = -2e^t$$

$$x = -2e^t(1-x) \to x - 2e^t x = -2e^t \to (1-2e^t)x = -2e^t$$

$$x = \frac{-2e^t}{1-2e^t} = \frac{-2}{-2+e^{-t}} = \frac{2}{2-e^{-t}}$$

4. Separate variables and use partial fraction to solve the initial value problem

$$\frac{dx}{dt} = 3x(5-x), x(0) = 8$$

$$\frac{1}{x(5-x)}dx = 3dt$$

$$\int \frac{1}{x(5-x)}dx = \int 3dt + C$$

Partial fraction

$$\frac{1}{5}\int (\frac{1}{x} + \frac{1}{5-x})dx = 3t + C$$

$$\ln(x) - \ln(5-x) = 15t + C$$

$$\ln\frac{x}{5-x} = 15t + C \to \frac{x}{5-x} = Ce^{15t}$$

$$\frac{8}{5-8} = Ce^{15(0)} = C \to C = -\frac{8}{3}$$

$$\frac{x}{5-x} = -\frac{8}{3}e^{15t}$$

$$-3x = (5-x)8e^{15t} \to -3x + 8xe^{15t} = 40e^{15t}$$

$$x(8e^{15t} - 3) = 40e^{15t} \to x(t) = \frac{40e^{15t}}{8e^{15t} - 3} = \frac{40}{8 - 3e^{-15t}}$$

5. Separate variables and use partial fraction to solve the initial value problem

$$\frac{dx}{dt} = 3x(x-5), x(0) = 2$$

$$\frac{1}{x(x-5)}dx = 3dt$$

$$\int \frac{1}{x(x-5)}dx = \int 3dt + C$$

Partial fraction

$$\frac{1}{x(x-5)} = \frac{A}{x} + \frac{B}{x-5} = \frac{A(x-5)+Bx}{x(x-5)} = \frac{-5A+(A+B)x}{x(x-5)}$$

$$-5A = 1, A + B = 0 \to A = -\frac{1}{5}, B = \frac{1}{5}$$

$$\frac{1}{x(x-5)} = \frac{-\frac{1}{5}}{x} + \frac{\frac{1}{5}}{x-5} = -\frac{1}{5}(\frac{1}{x} - \frac{1}{x-5})$$

Thus

$$\int \frac{1}{x(x-5)}dx = \int 3dt + C$$

becomes

$$-\frac{1}{5}\int (\frac{1}{x} - \frac{1}{x-5})dx = 3t + C$$

$$\ln(x) - \ln(x-5) = -15t + C \to \ln\frac{x}{x-5} = -15t + C$$

$$\frac{x}{x-5} = Ce^{-15t}$$

$$\frac{2}{2-5} = Ce^{-15(0)} = C \to C = -\frac{2}{3}$$

$$\frac{x}{x-5} = -\frac{2}{3}e^{-15t}$$

$$3x = -2(x-5)e^{-15t} \to 3x + 2xe^{-15t} = 10e^{-15t}$$

$$x = \frac{10e^{-15t}}{3+2e^{-15t}} = \frac{10}{2+3e^{15t}}$$

6. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to produce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is $k\frac{1}{\sqrt{P}}$. If there were initially 900 fish ($P(0) = 900$) in the lake and 441 were left after 6 weeks ($P(6) = 441$), how long did it take all the fish in the lake to die?

   Birth rate : $\beta(t) = 0$, Death rate : $\delta(t) = k\frac{1}{\sqrt{P}}$

$$\frac{dP}{dt} = (0 - k\frac{1}{\sqrt{P}})P, P(0) = 900$$

$$\frac{dP}{dt} = -k\sqrt{P}$$

$$\frac{1}{\sqrt{P}}dP = -kdt$$

$$\int \frac{1}{\sqrt{P}}dP = \int -kdt + C$$

$$2\sqrt{P} = -kt + C$$

$$2\sqrt{900} = -k(0) + C \rightarrow C = 60$$

$$2\sqrt{P} = -kt + 60$$

$$2\sqrt{441} = -6k + 60 \rightarrow 42 = -6k + 60 \rightarrow k = 3$$

$$2\sqrt{P} = -3t + 60$$

$$P = 0$$

$$0 = -3t + 60 \rightarrow t = 20 \text{ weeks}$$

7. Suppose that the number of alligator $x(t)$ (with t in months) of alligators in a swamp satisfies the differential equation

$$\frac{dx}{dt} = 0.0001x^2 - 0.01x$$

(a) If initially there are 25 alligators in the swamp, solve the differential equation to determine what happens to the alligator population in the long run ($t \to \infty$)

$$\frac{dx}{dt} = x(0.0001x - 0.01) = 0.0001x(x - 100), \quad x(0) = 25$$

$$\frac{1}{x(x-100)}dx = 0.0001dt$$

By integrating both sides

$$(\frac{1}{x} - \frac{1}{x-100})dx = -0.01dt$$

$$\ln \frac{x}{x-100} = -0.01t + C$$

$$\to \frac{x}{x-100} = Ce^{-0.01t}$$

Since $x(0) = 25$ (when $t = 0, x = 25$)

$$\frac{25}{25-100} = Ce^{-0.01(0)}$$

$$\frac{25}{-75} = C \to C = -\frac{1}{3}$$

$$\frac{x}{x-100} = -\frac{e^{-0.01t}}{3}$$

$$-3x = (x - 100)e^{-0.01t}$$

$$3x + xe^{-0.01t} = 100e^{-0.01t}$$

$$x(t) = \frac{100e^{-0.01t}}{3 + e^{-0.01t}} = \frac{100}{3e^{0.01t} + 1}$$

$$x(t) \to 0 \text{ as } t \to \infty$$

(b) Repeat part (a), except with 150 alligators initially.

$$\ln \frac{x}{x-100} = -0.01t + C \to \frac{x}{x-100} = Ce^{-0.01t}$$

Since $x(0) = 150$ (when $t = 0, x = 150$)

$$\frac{150}{150-100} = Ce^{-0.01(0)} \to C = 3$$

$$\frac{x}{x-100} = 3e^{-0.01t}$$

$$x = 3(x-100)e^{-0.01t}$$

$$x - 3xe^{-0.01t} = -300e^{-0.01t}$$

$$x = \frac{-300e^{-0.01t}}{1-3e^{-0.01t}}$$

When

$$1 - 3e^{-0.01t} = 0 \to e^{-0.01t} = \frac{1}{3} \to t = -\frac{\ln \frac{1}{3}}{0.01} \approx 109.8612 \text{ months}$$

the population explodes.