Differential Equations
Homework 4
Due Feb. 12, 2024 (**Monday**)

Note:

- Please show all of your work (writing a list of answers is not sufficient).
- Please indicate the people you worked with.
- **Please staple your HW.**
- Several random problems will be graded (1 point each).

1. Find the general solution of

$$x^2y' + 2xy = 5y^3$$
$$y' + \frac{2}{x}y = \frac{5}{x^2}y^3$$
$$\frac{1}{y^3}y' + \frac{2}{x}\frac{y}{y^3} = \frac{5}{x^2}$$
$$y^{-3}y' + \frac{2}{x}y^{-2} = \frac{5}{x^2}$$

This is a Bernoulli Eq. Let

$$v = y^{-2} \to v' = -2y^{-3}y' \to y' = -\frac{1}{2}y^3v'$$

Then, the differential equation becomes

$$y^{-3}(-\frac{1}{2}y^3v') + \frac{2}{x}v = \frac{5}{x^2}$$
$$(-\frac{1}{2}v') + \frac{2}{x}v = \frac{5}{x^2}$$
$$v' - \frac{4}{x}v = -\frac{10}{x^2}$$

By integrating factor method

$$v = \frac{1}{\mu}(\int -\frac{10}{x^2}\mu dx + C)$$

where

$$\mu = e^{\int -\frac{4}{x}dx} = x^{-4}$$

Thus,

$$v = x^4(\int -\frac{10}{x^2}x^{-4}dx + C) = x^4(\int -10x^{-6}dx + C) = x^4(\frac{10}{5}x^{-5} + C) = 2x^{-1} + Cx^4$$

$$y^{-2} = 2x^{-1} + Cx^4$$

or

$$y^2 = \frac{1}{2x^{-1} + Cx^4} = \frac{x}{2 + Cx^5}$$

or

$$y = \pm\sqrt{\frac{x}{2 + Cx^5}}$$

2. Find the general solution of

$$x^2y' + 2xy = 5y^4$$

$$y' + \frac{2}{x}y = \frac{5}{x^2}y^4$$

$$y^{-4}y' + \frac{2}{x}y^{-3} = \frac{5}{x^2}$$

This is a Bernoulli Eq. Let

$$v = y^{-3} \to v' = -3y^{-4}y' \to y' = -\frac{1}{3}y^4v'$$

Then, the differential equation becomes

$$y^{-4}(-\frac{1}{3}y^4v') + \frac{2}{x}v = \frac{5}{x^2}$$

$$(-\frac{1}{3}v') + \frac{2}{x}v = \frac{5}{x^2}$$

$$v' - \frac{6}{x}v = -\frac{15}{x^2}$$

By integrating factor method

$$v = \frac{1}{\mu}(\int -\frac{15}{x^2}\mu dx + C)$$

where

$$\mu = e^{\int -\frac{6}{x}dx} = x^{-6}$$

Therefore,

$$v = \frac{1}{x^{-6}}(\int -\frac{15}{x^2}x^{-6}dx + C) = x^6(\int -15x^{-8}dx + C)$$

$$= x^6(\frac{15}{7}x^{-7} + C) = \frac{15}{7}x^{-1} + Cx^6$$

$$y^{-3} = \frac{15}{7}x^{-1} + Cx^6 \to y = \sqrt[3]{\frac{1}{\frac{15}{7}x^{-1} + Cx^6}} = \sqrt[3]{\frac{7x}{15 + 7Cx^7}}$$

3. Find the general solution of

$$xy' + 6y = 3xy^{4/3}$$
$$y' + \frac{6}{x}y = 3y^{4/3}$$
$$y^{-4/3}y' + \frac{6}{x}y^{-1/3} = 3$$

This is a Bernoulli Eq. Let

$$v = y^{-\frac{1}{3}} \to v' = -\frac{1}{3}y^{-\frac{4}{3}}y' \to y' = -3y^{\frac{4}{3}}v'$$

Then, the differential equation becomes

$$y^{-4/3}(-3y^{\frac{4}{3}}v') + \frac{6}{x}v = 3$$
$$(-3v') + \frac{6}{x}v = 3$$
$$v' - \frac{2}{x}v = -1$$

By integrating factor method

$$v = \frac{1}{\mu}(\int -1\mu dx + C)$$

where

$$\mu = e^{\int -\frac{2}{x}dx} = x^{-2}$$
$$v = \frac{1}{x^{-2}}(\int -x^{-2}dx + C) = x^2(\frac{1}{x} + C) = x + Cx^2$$
$$y^{-\frac{1}{3}} = x + Cx^2$$
$$y = (x + Cx^2)^{-3}$$

4. Verify that the given differential equation is exact; then solve it

$$(4x - y)dx + (6y - x)dy = 0$$
$$M = 4x - y, N = 6y - x$$
$$\frac{\partial}{\partial y}M = -1 = \frac{\partial}{\partial x}N = -1, \to \text{Exact Eq}$$

Thus

$$\frac{\partial F}{\partial x} = 4x - y \to F = 2x^2 - xy + g(y) \to \frac{\partial F}{\partial y} = -x + g'(y)$$

$$\frac{\partial F}{\partial y} = 6y - x$$

$$g'(y) = 6y \to g(y) = 3y^2 + D$$

Therefore,

$$F(x, y) = 2x^2 - xy + 3y^2 + D$$

and the solution of the differential equation is

$$2x^2 - xy + 3y^2 + D = C \to 2x^2 - xy + 3y^2 = C$$

5. Verify that the given differential equation is exact; then solve it

$$(x^3 + \frac{y}{x})dx + (y^2 + \ln x)dy = 0$$

$$M = x^3 + \frac{y}{x}, N = y^2 + \ln x$$

$$\frac{\partial}{\partial y}M = \frac{1}{x} = \frac{\partial}{\partial x}N = \frac{1}{x}, \to \text{Exact Eq}$$

Thus

$$\frac{\partial F}{\partial x} = x^3 + \frac{y}{x} \to F = \frac{1}{4}x^4 + y\ln x + g(y) \to \frac{\partial F}{\partial y} = \ln x + g'(y)$$

$$\frac{\partial F}{\partial y} = y^2 + \ln x$$

$$g'(y) = y^2 \to g(y) = \frac{1}{3}y^3 + D$$

Therefore,

$$F(x, y) = \frac{1}{4}x^4 + y\ln x + \frac{1}{3}y^3$$

and the solution of the differential equation is

$$\frac{1}{4}x^4 + y\ln x + \frac{1}{3}y^3 + D = C \to \frac{1}{4}x^4 + y\ln x + \frac{1}{3}y^3 = C$$

or

$$3x^4 + 12y\ln x + 4y^3 = C$$

6. Verify that the given differential equation is exact; then solve it

$$(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$$

$$M = 1 + ye^{xy}, N = 2y + xe^{xy}$$

$$\frac{\partial}{\partial y}M = e^{xy} + xye^{xy} = \frac{\partial}{\partial x}N = e^{xy} + xye^{xy}, \rightarrow \text{Exact Eq}$$

Thus

$$\frac{\partial F}{\partial x} = 1 + ye^{xy} \rightarrow F = x + e^{xy} + g(y) \rightarrow \frac{\partial F}{\partial y} = xe^{xy} + g'(y)$$

$$\frac{\partial F}{\partial y} = 2y + xe^{xy}$$

$$g'(y) = 2y \rightarrow g(y) = y^2 + D$$

Therefore,

$$F(x, y) = x + e^{xy} + y^2 + D$$

and the solution of the differential equation is

$$x + e^{xy} + y^2 + D = C \rightarrow x + e^{xy} + y^2 = C$$

7. Verify that the given differential equation is exact; then solve it

$$(\cos x + \ln y)dx + (\frac{x}{y} + e^y)dy = 0$$

$$M = \cos x + \ln y, N = \frac{x}{y} + e^y$$

$$\frac{\partial}{\partial y}M = \frac{1}{y} = \frac{\partial}{\partial x}N = \frac{1}{y}, \rightarrow \text{Exact Eq}$$

Thus

$$\frac{\partial F}{\partial x} = \cos x + \ln y \rightarrow F = \sin x + x \ln y + g(y) \rightarrow \frac{\partial F}{\partial y} = \frac{x}{y} + g'(y)$$

$$\frac{\partial F}{\partial y} = \frac{x}{y} + e^y$$

$$g'(y) = e^y \rightarrow g(y) = e^y + D$$

Therefore,

$$F(x, y) = \sin x + x \ln y + e^y + D$$

and the solution of the differential equation is

$$\sin x + x \ln y + e^y + D = C \rightarrow \sin x + x \ln y + e^y = C$$

8. Show that the substitution $v = \ln y$ transforms

$$\frac{dy}{dx} + P(x)y = Q(x)(y \ln y)$$

into the linear differential equation

$$\frac{dv}{dx} + P(x) = Q(x)v(x).$$

Proof: Let

$$v = \ln y \to \frac{dv}{dx} = \frac{1}{y}\frac{dy}{dx} \to \frac{dy}{dx} = y\frac{dv}{dx}$$

Then, the differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)(y \ln y)$$

becomes

$$y\frac{dv}{dx} + P(x)y = Q(x)(yv(x))$$

Dividing both sides by y yields

$$\frac{dv}{dx} + P(x) = Q(x)v(x)$$