Differential Equations
Homework 3
Due Feb. 7, 2024 (Wed.)

Note:

- Please show all of your work (writing a list of answers is not sufficient).
- Please indicate the people you worked with.
- **Please staple your HW.**
- Several random problems will be graded (1 point each).

1. Find the general solution

$$y' + 3y = 2xe^{-3x}$$

$$\mu(y' + 3y) = \mu(2xe^{-3x}) \to \mu y' + 3\mu y = 2\mu xe^{-3x}$$

If

$$\mu' = 3\mu$$

The differential equation becomes

$$\mu y' + \mu' y = 2\mu xe^{-3x}$$

Then, by the product rule

$$(\mu y)' = 2\mu xe^{-3x} \to \mu y = \int 2\mu xe^{-3x}dx + C$$

$$y = \frac{1}{\mu}(\int 2\mu xe^{-3x}dx + C)$$

From

$$\mu' = 3\mu$$

$$\frac{d\mu}{dx} = 3\mu \to \frac{1}{\mu}d\mu = 3dx \to \int \frac{1}{\mu}d\mu = \int 3dx$$

$$\ln|\mu| = 3x \to |\mu| = e^{3x} \to \mu = e^{3x}$$

Therefore,

$$y = \frac{1}{e^{3x}}(\int 2e^{3x}xe^{-3x}dx + C) = e^{-3x}(\int 2xdx + C) = e^{-3x}(x^2 + C) = x^2e^{-3x} + Ce^{-3x}$$

2. Find the particular solution of the initial value problem

$$xy' - y = x, y(1) = 7$$
$$y' - \frac{1}{x}y = 1, y(1) = 7$$

Please see #3 for derivation

$$y = \frac{1}{\mu}(\int \mu \cdot 1dx + C),$$

where

$$\mu = e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = x^{-1}$$

Therefore,

$$y = \frac{1}{x^{-1}}(\int x^{-1}dx + C) = x(\ln x + C),$$
$$y(1) = 1(\ln 1 + C) = C = 7$$
$$y(x) = x(\ln x + 7)$$

3. Find the particular solution of the initial value problem

$$y' + y = e^x, y(0) = 1$$

Please see #3 for derivation

$$y = \frac{1}{\mu}(\int \mu e^x dx + C),$$

where

$$\mu = e^{\int 1dx} = e^x$$

Therefore,

$$y = \frac{1}{e^x}(\int e^x e^x dx + C) = e^{-x}(\int e^{2x}dx + C) = e^{-x}(\frac{1}{2}e^{2x} + C)$$
$$y = \frac{1}{2}e^x + Ce^{-x}$$
$$y(0) = \frac{1}{2}e^0 + Ce^{-0} = \frac{1}{2} + C = 1 \rightarrow C = \frac{1}{2}$$
$$y(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$$

4. Find the particular solution of the initial value problem

$$(1+x)y' + y = \cos x, y(0) = 1$$

$$y' + \frac{1}{1+x}y = \frac{\cos x}{1+x}$$

Please see #3 for derivation

$$y = \frac{1}{\mu}(\int \mu \frac{\cos x}{1+x}dx + C),$$

where

$$\mu = e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1+x$$

Therefore,

$$y = \frac{1}{1+x}(\int (1+x)\frac{\cos x}{1+x}dx + C) = \frac{1}{1+x}(\int \cos x dx + C)$$

$$y = \frac{1}{1+x}\sin x + \frac{C}{1+x}$$

$$y(0) = \frac{1}{1+0}\sin 0 + \frac{C}{1+0} = C = 1$$

$$y = \frac{1}{1+x}\sin x + \frac{1}{1+x} = \frac{1+\sin x}{1+x}$$

5. Find the particular solution of the initial value problem

$$y' = 1 + x + y + xy, y(0) = 0$$

$$y' - (1+x)y = 1+x$$

Please see #3 for derivation

$$y = \frac{1}{\mu}(\int \mu(1+x)dx + C),$$

where

$$\mu = e^{\int -(1+x)dx} = e^{-x-\frac{1}{2}x^2}$$

Therefore,

$$y = \frac{1}{e^{-x-\frac{1}{2}x^2}}(\int e^{-x-\frac{1}{2}x^2}(1+x)dx + C),$$

Integrate using substitution $u = -x - \frac{1}{2}x^2$, then

$$y = -1 + Ce^{x+\frac{1}{2}x^2}$$

$$y(0) = -1 + C = 0 \rightarrow C = 1$$

$$y = -1 + e^{x+\frac{1}{2}x^2}$$

6. Find the particular solution of the initial value problem

$$(x^2+4)y' + 3xy = x, y(0) = 1$$

$$y' + \frac{3x}{(x^2+4)}y = \frac{x}{(x^2+4)}$$

Please see #3 for derivation

$$y = \frac{1}{\mu}(\int \mu \frac{x}{(x^2+4)}dx + C),$$

where

$$\mu = e^{\int \frac{3x}{(x^2+4)}dx} = (x^2+4)^{3/2}$$

Therefore,

$$y = \frac{1}{(x^2+4)^{3/2}}(\int (x^2+4)^{3/2}\frac{x}{(x^2+4)}dx + C),$$

$$= \frac{1}{(x^2+4)^{3/2}}(\int x(x^2+4)^{1/2}dx + C) = \frac{1}{(x^2+4)^{3/2}}(\frac{1}{2}\int \sqrt{u}du + C)$$

where $u = x^2 + 4$

$$= \frac{1}{(x^2+4)^{3/2}}(\frac{1}{3}(u)^{3/2} + C) = \frac{1}{(x^2+4)^{3/2}}(\frac{1}{3}(x^2+4)^{3/2} + C)$$

$$= \frac{1}{3} + C(x^2+4)^{-3/2}$$

$$y(0) = \frac{1}{3} + C(4)^{-\frac{3}{2}} = \frac{1}{3} + \frac{C}{8} = 1 \rightarrow C = \frac{16}{3}$$

$$y(x) = \frac{1}{3} + \frac{16}{3}(x^2+4)^{-3/2}$$

7. A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture – kept uniform by stirring – is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

$$c_i = 0, r_i = 5$$

$$c_o = \frac{x(t)}{1000 + (5-5)t} = \frac{x(t)}{1000}, r_o = 5$$

$$\frac{dx}{dt} = 0 \times 5 - \frac{x(t)}{1000} \times 5 = -\frac{x(t)}{200}, x(0) = 100$$

$$\frac{dx}{dt} = -\frac{x}{200}, x(0) = 100$$

$$\frac{1}{x(t)}dx = -\frac{1}{200}dt$$

$$\int \frac{1}{x(t)}dx = \int -\frac{1}{200}dt + C$$

$$\ln x = -\frac{1}{200}t + C$$

$$x = e^{-\frac{1}{200}t+C} = e^C e^{-\frac{1}{200}t} = Ce^{-\frac{1}{200}t}$$

$$x(0) = C = 100$$

$$x(t) = 100e^{-\frac{1}{200}t}$$

$$10 = 100e^{-\frac{1}{200}t} \to 0.1 = e^{-\frac{1}{200}t} \to -\frac{1}{200}t = \ln 0.1 \to t = -200\ln 0.1 = 460.52$$

8. A tank initially contains 60 gal of pure water. Brine consisting of 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min: thus the tank is empty after exactly 1 h.

   (a) Find the amount of salt in the tank after t minutes.

   (b) What is the maximum amount of salt ever in the tank?

$$c_i = 1, r_i = 2$$

$$c_o = \frac{x(t)}{60 + (2-3)t} = \frac{x(t)}{60-t}, r_o = 3$$

$$\frac{dx}{dt} = 1 \times 2 - \frac{x(t)}{60-t} \times 3 = 2 - \frac{3x}{60-t}, x(0) = 0$$

$$\frac{dx}{dt} = 2 - \frac{3x}{60-t}, x(0) = 0$$

$$\frac{dx}{dt} + \frac{3}{60-t}x = 2$$

$$x(t) = \frac{1}{\mu}(\int \mu 2dt + C)$$

where

$$\mu = e^{\int \frac{3}{60-t}dt} = e^{-3\ln(60-t)} = \frac{1}{(60-t)^3}$$

Therefore

$$x(t) = (60-t)^3\left(\int \frac{2}{(60-t)^3}dt + C\right) = (60-t)^3\left(\frac{1}{(60-t)^2} + C\right)$$

$$x(t) = (60-t) + C(60-t)^3$$

$$x(0) = 60 + C(60)^3 = 0 \rightarrow C = -\frac{1}{60^2}$$

$$x(t) = (60-t) - \frac{1}{60^2}(60-t)^3$$

$$x'(t) = -1 + \frac{3}{60^2}(60-t)^2 = 0 \rightarrow (60-t)^2 = \frac{60^2}{3} \rightarrow 60-t = \frac{60}{\sqrt{3}} \rightarrow t = 60 - \frac{60}{\sqrt{3}} \approx 25.36$$

The maximum amount of salt ever in the tank is

$$x(25.36) \approx 23.0940$$