Differential Equations
Homework 2
Due Jan. 31, 2024 (Wed.)

Note:

- Please show all of your work (writing a list of answers is not sufficient).
- Please indicate the people you worked with.
- Several random problems will be graded (1 point each).

1. Find a function $y = f(x)$ satisfying

$$\frac{dy}{dx} = (x-2)^2; y(2) = 1$$

$$y(x) = \int (x-2)^2 dx + C = \frac{1}{3}(x-2)^3 + C$$

$$y(2) = \frac{1}{3}(3-3)^2 + C = 1 \rightarrow C = 1$$

$$y(x) = \frac{1}{3}(x-2)^3 + 1$$

2. Find a function $y = f(x)$ satisfying

$$\frac{dy}{dx} = x\sqrt{x^2+9}; y(-4) = 0$$

$$y(x) = \int x\sqrt{x^2+9}dx + C$$

Let

$$u = x^2 + 9 \rightarrow du = 2xdx \rightarrow dx = \frac{1}{2x}du$$

Then

$$y(x) = \int x\sqrt{x^2+9}dx + C = \int x\sqrt{u}\frac{1}{2x}du + C = \frac{1}{2}\int \sqrt{u}du + C$$

$$= \frac{1}{2}\int u^{1/2}du + C = \frac{1}{2}\frac{2}{3}u^{3/2} + C = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+9)^{3/2} + C$$

$$y(-4) = \frac{1}{3}(16+9)^{(3/2)} + C = 0 \rightarrow C = -\frac{125}{3}$$

$$y(x) = \frac{1}{3}(x^2+9)^{3/2} - \frac{125}{3}$$

3. Find a function $y = f(x)$ satisfying

$$\frac{dy}{dx} = xe^{-x}; y(0) = 1$$

$$y(x) = \int xe^{-x} dx + C$$

Integration by parts

$$y(x) = -xe^{-x} - \int -e^{-x} dx + C = -xe^{-x} + \int e^{-x} dx + C = -xe^{-x} - e^{-x} + C$$

$$y(0) = -0e^{-0} - e^{-0} + C = 1 \rightarrow -1 + C = 1 \rightarrow C = 2$$

$$y(x) = -xe^{-x} - e^{-x} + 2$$

4. A diesel car gradually speeds up so that for the first 10 s its acceleration is given by

$$\frac{dv}{dt} = 0.12t^2 + 0.6t \quad (ft/s^2).$$

If the car starts from rest ($x_0 = 0$, $v_0 = 0$), find the distance it has traveled at the end of the first 10 s and its velocity at that time

$$\frac{dv}{dt} = 0.12t^2 + 0.6t$$

$$x(0) = 0, v(0) = 0$$

$$v(t) = 0.04t^3 + 0.3t^2 + C$$

$$v(0) = C = 0$$

$$v(t) = 0.04t^3 + 0.3t^2$$

$$\frac{dx}{dt} = 0.04t^3 + 0.3t^2$$

$$x(t) = 0.01t^4 + 0.1t^3 + C$$

$$x(0) = C = 0$$

$$x(t) = 0.01t^4 + 0.1t^3$$

$$x(10) = 200, v(10) = 70$$

5. Find the general solution of

$$\frac{dy}{dx} + 2xy = 0$$

$$\frac{dy}{dx} = -2xy \to \frac{1}{y}dy = -2xdx$$

$$\int \frac{1}{y}dy = \int -2xdx + C \to \ln|y| = -x^2 + C \to |y| = e^{-x^2+C} = e^{-x^2}e^C = Ce^{-x^2}$$

$$y = Ce^{-x^2}$$

6. Find the general solution of

$$(1+x)^2\frac{dy}{dx} = (1+y)^2$$

$$\frac{1}{(1+y)^2}dy = \frac{1}{(1+x)^2}dx \to \int \frac{1}{(1+y)^2}dy = \int \frac{1}{(1+x)^2} + C$$

$$-\frac{1}{(1+y)} = -\frac{1}{(1+x)} + C = \frac{-1 + C(1+x)}{1+x}$$

$$(1+y)(1 - C(1+x)) = 1 + x$$

$$(1+y) = \frac{1+x}{(1 - C(1+x))}$$

$$y = \frac{1+x}{(1 - C(1+x))} - 1$$

or

$$y = \frac{1+x}{(1 + C(1+x))} - 1$$

7. Find the general solution of

$$y^3\frac{dy}{dx} = (y^4 + 1)\cos x$$

$$\frac{y^3}{y^4+1}dy = \cos x dx$$

$$\int \frac{y^3}{y^4+1}dy = \int \cos x dx + C$$

By using substitution ($u = y^4 + 1$)

$$\frac{1}{4}\ln(y^4 + 1) = \sin x + C$$

$$\ln(y^4+1) = 4\sin x + C$$

or

$$y^4 + 1 = e^{4\sin x + C} = e^C e^{4\sin x} = Ce^{4\sin x}$$
$$y^4 = Ce^{4\sin x} - 1$$

8. Find explicit particular solution of

$$\frac{dy}{dx} = 2xy^2 + 3x^2y^2, y(1) = -1$$

$$\frac{dy}{dx} = 2xy^2 + 3x^2y^2 = (2x+3x^2)y^2 \to \frac{1}{y^2}dy = (2x+3x^2)dx$$
$$\int \frac{1}{y^2}dy = \int (2x+3x^2)dx + C \to -\frac{1}{y} = x^2 + x^3 + C$$
$$-\frac{1}{-1} = 1^2 + 1^3 + C \to C = -1$$
$$-\frac{1}{y} = x^2 + x^3 - 1 \to y = -\frac{1}{x^2+x^3-1}$$

9. In a certain culture of bacteria, the number of bacteria increased sixfold in 10 h. How long did it take for the population to double?

$$P(0) = P_0$$
$$P(10) = 6P_0$$
$$P(t) = 2P_0 \to t = ?$$
$$\frac{dP}{dt} = kP, P(0) = P_0$$
$$\frac{1}{P}dP = kdt \to \int \frac{1}{P}dP = \int kdt + C$$
$$\ln|P| = kt + C \to |P| = e^{kt+C} = e^C e^{kt} = Ce^{kt}$$
$$P = \pm Ce^{kt} = De^{kt}$$
$$P(0) = De^{k\cdot 0} = D = P_0$$
$$P(t) = P_0 e^{kt}$$
$$P(10) = P_0 e^{10k} = 6P_0 \to e^{10k} = 6 \to k = \frac{\ln 6}{10}$$
$$P(t) = P_0 e^{\frac{\ln 6}{10}t}$$
$$P(t) = P_0 e^{\frac{\ln 6}{10}t} = 2P_0 \to e^{\frac{\ln 6}{10}t} = 2 \to t = \frac{\ln 2}{\frac{\ln 6}{10}} \approx 3.8685\ h$$