

Title: Antiderivatives

Goal:

- Define **antiderivative** and **most general antiderivative**
- Find basic antiderivatives
- Define **indefinite integral**

Lecture:

Recall from Calculus I that if $s(t)$ represents an object's position at time t , then the object's velocity function, $v(t)$, is given by $v(t) = s'(t)$.

Suppose you were given the velocity function. Could you find the position function? In other words, if you were given a function v , could you find a function s whose derivative is equal to v ?

Hopefully you can see that it would be very useful that if given a function f , you were able to find a function F whose derivative is equal to f .

Definition:

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I . ■

Example 1:

Find the antiderivative of $f(x) = 5x^4$.

Solution:

Here, $f(x)$ is a constant times a power function. We know that the derivative of a power function is $\frac{d}{dx}(x^n) = nx^{n-1}$ – that is, we decrease the exponent by one. To find the antiderivative, we are going “backwards”, so we will try increasing the exponent by one. Let's try $F(x) = x^5$. Notice that $F'(x) = 5x^4$; that is $F'(x) = f(x)$. So $F(x) = x^5$ is an antiderivative of $f(x) = 5x^4$. ■

About now, you should be thinking “but $F(x) = x^5 + 1$ is also an antiderivative of $f(x) = 5x^4$...how will I know if I have found ALL of the antiderivatives of $f(x)$?”

Well, back in Calculus I, you saw a Theorem called the Mean Value Theorem and with it a very important Corollary. The Corollary states:

If $F'(x) = G'(x)$ at each point x in an open interval (a, b) , then there exists a constant C such that $f(x) = g(x) + C$ for all x in (a, b) .

What this means to us is that once we find one antiderivative of $f(x)$, then we know all of them! This leads to the following definition:

Definition:

If F is an antiderivative of f on an interval I , then the **most general antiderivative** of f on I is

$$F(x) + C$$

where C is an arbitrary constant. ■

The most general antiderivative is a family of functions. We use this concept so much that we will make a new definition and add some notation:

Definition:

The most general antiderivative of f is called the **indefinite integral** of f with respect to x , and is denoted by

$$\int f(x) dx.$$

The symbol \int is an **integral sign**. The function f is the **integrand** of the indefinite integral, and x is the **variable of integration**. ■

Example 2:

$$\int \cos x dx = \sin x + C \quad \text{because} \quad \frac{d}{dx}(\sin x) = \cos x$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C \quad \text{because} \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\int e^x dx = e^x + C \quad \text{because} \quad \frac{d}{dx}(e^x) = e^x \quad \blacksquare$$

You will want to understand and memorize the indefinite integrals given on the next page. Since you already know the derivatives of the basic functions (power, exponential, logarithmic, trigonometric, and inverse trigonometric functions), memorization should be easy.

Basic Indefinite Integrals Table

k is a non-zero constant

$$\int k \, dx = kx + C$$

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \text{ where } n \neq -1$$

$$\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C$$

$$\int a^{kx} \, dx = \frac{1}{k} \cdot \frac{1}{\ln a} a^{kx} + C, \text{ where } a > 0, a \neq 1$$

$$\int \frac{1}{kx} \, dx = \frac{1}{k} \ln|kx| + C = \frac{1}{k} \ln|x| + C$$

$$\int \cos(kx) \, dx = \frac{1}{k} \sin(kx) + C$$

$$\int \sin(kx) \, dx = -\frac{1}{k} \cos(kx) + C$$

$$\int \sec^2(kx) \, dx = \frac{1}{k} \tan(kx) + C$$

$$\int \csc(kx) \cot(kx) \, dx = -\frac{1}{k} \csc(kx) + C$$

$$\int \sec(kx) \tan(kx) \, dx = \frac{1}{k} \sec(kx) + C$$

$$\int \csc^2(kx) \, dx = -\frac{1}{k} \cot(kx) + C$$

$$\int \frac{1}{\sqrt{1-(kx)^2}} \, dx = \frac{1}{k} \sin^{-1}(kx) + C$$

$$\int \frac{1}{1+(kx)^2} \, dx = \frac{1}{k} \tan^{-1}(kx) + C$$

$$\int \frac{1}{(kx)\sqrt{(kx)^2-1}} \, dx = \frac{1}{k} \sec^{-1}|kx| + C$$

Once you know these, you can find all kinds of indefinite integrals by using the following properties of indefinite integrals.

Properties of Indefinite Integrals

k is a non-zero constant

Constant Multiple Rule $\int kf(x) dx = k \int f(x) dx$

Negative Rule $\int(-f(x)) dx = - \int f(x) dx$

Sum Rule $\int(f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$

Difference Rule $\int(f(x) - g(x)) dx = \int f(x) dx - \int g(x) dx$

Example 3:

Find the indefinite integrals. You may need to try a solution and then adjust your guess. Also, you may need to rewrite the integrand in an equivalent but different way before you can find the antiderivatives.

a) $\int x^7 dx$

Solution to a):

Using the Basic Indefinite Integrals Table with $n = 7$, we get

$$\begin{aligned}\int x^7 dx &= \frac{1}{7+1} x^{7+1} + C \\ &= \frac{1}{8} x^8 + C \blacksquare\end{aligned}$$

b) $\int \cos(5x) dx$

Solution to b):

Using the Basic Indefinite Integrals Table with $k = 5$, we get

$$\int \cos(5x) dx = \frac{1}{5} \sin(5x) + C \blacksquare$$

c) $\int 3 \cos(5x) dx$

Solution to c):

Using the Constant Multiple Rule we get:

$$\begin{aligned}\int 3 \cos(5x) dx &= 3 \int \cos(5x) dx \\ &= 3 \cdot \left(\frac{1}{5} \sin(5x) + C_1\right) \\ &= \frac{3}{5} \sin(5x) + C \blacksquare\end{aligned}$$

Since C_1 is an arbitrary constant, $3C_1$ is also an arbitrary constant and we can just write it as C .

d) $\int (3 \cos(5x) + x^7) dx$

Solution to d):

Using the Addition Rule we get:

$$\begin{aligned}\int (3 \cos(5x) + x^7) dx &= \int 3 \cos(5x) dx + \int x^7 dx \\ &= \frac{3}{5} \sin(5x) + C_1 + \frac{1}{8} x^8 + C_2 \\ &= \frac{3}{5} \sin(5x) + \frac{1}{8} x^8 + C \blacksquare\end{aligned}$$

Since C_1 and C_2 are arbitrary constants, $C_1 + C_2$ is also an arbitrary constant and we can just write it as C .

e) $\int (3x + 1)^4 dx$

Solution to e):

Let's guess. The integrand is a composition of two functions and the outside function is a power function. We know that $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$, so let's guess:

$$G(x) = \frac{1}{5} (3x + 1)^5$$

Next we check by differentiating:

$$G'(x) = 3(3x + 1)^4 \quad \text{Make sure you can do this differentiation!}$$

We want $G'(x)$ to equal the integrand and it does not. So now we adjust:

$$F(x) = \frac{1}{3} G(x) = \frac{1}{3} \cdot \frac{1}{5} (3x + 1)^5 = \frac{1}{15} (3x + 1)^5$$

And this is great because

$$F'(x) = (3x + 1)^4$$

Again, make sure you can do this differentiation!

Therefore

$$\int (3x + 1)^4 dx = \frac{1}{15} (3x + 1)^5 + C \blacksquare$$

f) $\int x(x + 1)^2 dx$

Solution to f):

Here the integrand is a product of two functions: x and $(x + 1)^2$. We do NOT have any rules that tell us how to integrate the product of two functions. In order to solve this indefinite integral, we need to rewrite the integrand. Notice that:

$$x(x + 1)^2 = x(x^2 + 2x + 1) = x^3 + 2x^2 + x$$

Substituting this in we get:

$$\begin{aligned} \int x(x + 1)^2 dx &= \int (x^3 + 2x^2 + x) dx \\ &= \int x^3 dx + \int 2x^2 dx + \int x dx \\ &= \int x^3 dx + 2 \int x^2 dx + \int x dx \\ &= \frac{1}{4} x^4 + 2 \left(\frac{1}{3} x^3 \right) + \frac{1}{2} x^2 + C \\ &= \frac{1}{4} x^4 + \frac{2}{3} x^3 + \frac{1}{2} x^2 + C \blacksquare \end{aligned}$$

Very important comments about notation:

- Every “ \int ” must have a “ dx ”.
- If your integrand is the addition or subtraction of two or more functions, or if your integrand begins with a negative sign, then you must put the integrand in parenthesis.
- The “ dx ” can be placed in different locations; e.g. $\int \frac{f(x) dx}{g(x)} = \int \frac{f(x)}{g(x)} dx$
- $\int dx = \int 1 dx$

Example 4:

A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec². What is the velocity of the rocket at time t sec?

Solution:

Let $v(t)$ be the velocity (in m/sec) of the rocket at time t sec and $a(t)$ be the acceleration (in m/sec²) of the rocket at time t sec.

The problem tells us that $v(0) = 0$ (because it is lifting off) and that $a(t) = 20$.

Since $\frac{dv}{dt} = a$ we know that $v(t)$ is an antiderivative of $a(t)$.

$$\int a(t) dt = \int 20 dt = 20t + C$$

This means that $v(t) = 20t + C$. To find C , we will use the fact that $v(0) = 0$.

$$\begin{aligned} 0 &= v(0) \\ &= 20 \cdot 0 + C \\ &= C \end{aligned}$$

Since $C = 0$, we get:

$$v(t) = 20t + 0 = 20t \blacksquare$$

Summary:

1. A function F is an antiderivative of f if $F'(x) = f(x)$
2. If $F'(x) = f(x)$ then $\int f(x) dx = F(x) + C$
3. Basic Indefinite Integrals Table – memorize these
4. Properties of Indefinite Integrals
5. Notation is very important