Convergence of Taylor Series

Part 2

Using Taylor Series

Since every Taylor series is a power series, the operations of adding, subtracting, and multiplying Taylor series are all valid on the intersection of their intervals of convergence.

Example 1

Find the Taylor series at $x = 0$ of e^{-6x}.

Solution:

Since

$$e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

we get

$$e^{-6x} = \sum_{k=0}^{\infty} \frac{(-6x)^k}{k!}$$

Example 2

Find the Taylor series at $x = 0$ of $3xe^x$.

<u>Solution</u>:

Since

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

we get

$$3xe^x = 3x \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{3x^{k+1}}{k!}$$

Example 3

Find the Taylor series for $x^2 \cos \frac{\pi x}{2}$ at $x = 0$.

Solution:

Since

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

we get

Example 3 (continued)

$$\cos\frac{\pi x}{2} = \sum_{k=0}^{\infty}(-1)^k \frac{\left(\frac{\pi x}{2}\right)^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty}(-1)^k \left(\frac{\pi}{2}\right)^{2k} \frac{x^{2k}}{(2k)!}$$

Therefore

$$x^2 \cos\frac{\pi x}{2} = x^2 \sum_{k=0}^{\infty}(-1)^k \left(\frac{\pi}{2}\right)^{2k} \frac{x^{2k}}{(2k)!}$$

$$= \boxed{\sum_{k=0}^{\infty}(-1)^k \left(\frac{\pi}{2}\right)^{2k} \frac{x^{2k+2}}{(2k)!}}$$

Example 4

Find the Taylor series at $x = 0$ of $\frac{x^2}{1-x}$.

<u>Solution</u>:

Since

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

we get

$$\frac{x^2}{1-x} = x^2 \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} x^{k+2}$$

Example 5

Find the first four nonzero terms in the Maclaurin series for

$$f(x) = e^{-6x} \sin x \,.$$

<u>Solution</u>:

We will use the Taylor series for e^{-6x} and $\sin x$ and then use:

If $\sum a_n x^n$ and $\sum b_n x^n$ converge absolutely for $|x| < R$ and

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

then

$$\left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum c_n x^n$$

which also converges for $|x| < R$.

Example 5 (continued)

$$e^{-6x} = \sum_{k=0}^{\infty} \frac{(-6x)^k}{k!} = 1 - 6x + 18x^2 - 36x^3 + 54x^4 - \cdots$$

So we will let $a_0 = 1, a_1 = -6, a_2 = 18, a_3 = -36, a_4 = 54.$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$= 0 + x + 0 \cdot x^2 - \frac{1}{6}x^3 + 0 \cdot x^4 + \cdots$$

So we will let $b_0 = 0, b_1 = 1, b_2 = 0, b_3 = -\frac{1}{6}, b_4 = 0.$

Example 5 (continued)

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$$\begin{aligned}
c_0 &= a_0 b_0 \\
&= 1 \cdot 0 = 0 \\
c_1 &= a_0 b_1 + a_1 b_0 \\
&= 1 \cdot 1 + (-6) \cdot 0 = 1 \\
c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\
&= 1 \cdot 0 + (-6) \cdot 1 + 18 \cdot 0 = -6 \\
c_3 &= a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \\
&= 1 \cdot \left(-\frac{1}{6}\right) + (-6) \cdot 0 + 18 \cdot 1 + (-36) \cdot 0 = \frac{107}{6} \\
c_4 &= a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0 \\
&= 1 \cdot 0 + (-6)\left(-\frac{1}{6}\right) + 18 \cdot 0 + (-36) \cdot 1 + 54 \cdot 0 = -35
\end{aligned}$$

Example 5 (continued)

Therefore

$$e^{-6x}\sin x = \left(\sum_{k=0}^{\infty}\frac{(-6x)^k}{k!}\right)\left(\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\right)$$

$$= \sum_{k=0}^{\infty} c_n x^n$$

$$= x - 6x^2 + \frac{107}{6}x^3 - 35x^4 + \cdots$$

Example 6

Find the first four nonzero terms in the Maclaurin series for $\frac{3}{(1+x)^2}$.

<u>Solution</u>:

Since

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

we get

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k$$

Example 6 (continued)

Notice that

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{-1}{(1+x)^2}$$

and that

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}\left(\sum_{k=0}^{\infty}(-1)^k\, x^k\right)$$

$$= \sum_{k=0}^{\infty}\frac{d}{dx}\left((-1)^k x^k\right)$$

$$= \sum_{k=0}^{\infty}(-1)^k\, kx^{k-1}$$

Example 6 (continued)

Using these results we get:

$$\frac{3}{(1+x)^2} = -3\,\frac{-1}{(1+x)^2}$$

$$= -3\sum_{k=0}^{\infty}(-1)^k\,kx^{k-1}$$

$$= \sum_{k=0}^{\infty}(-1)^{k+1}\,3kx^{k-1}$$

$$= \boxed{3 - 6x + 9x^2 - 12x^3 + \cdots}$$

Why You Need to Know Taylor Series

Russian physicist Igor Tamm won the Nobel Prize in physics in 1958. During the Russian revolution, he was a physics professor at the University of Odessa in the Ukraine. Food was in short supply, so he made a trip to a nearby village in search of food. While he was in the village, a bunch of anti-communist bandits surrounded the town.

The leader was suspicious of Tamm, who was dressed in city clothes. He demanded to know what Tamm did for a living. He explained that he was a university professor looking for food. “What subject?,” the bandit leader asked. Tamm replied “I teach mathematics.”

“Mathematics?” said the leader. “OK. Then give me an estimate of the error one makes by cutting off a Maclaurin series expansion at the *n* th term. Do this and you will go free. Fail, and I will shoot you.”

Tamm was not just a little astonished. At gunpoint, he managed to work out the answer. He showed it to the bandit leader, who perused it and then declared “Correct! Go home.” Tamm never discovered the name of the bandit.

From “Calculus makes you live longer”, in “100 essential things you didn’t know you didn’t know”, by John Barrow.

http://math.stackexchange.com/questions/28885/anecdotes-about-famous-mathematicians-or-physicists