Power Series

Part 2

Differentiation & Integration;

Multiplication of Power Series

Theorem 1

If $\sum a_n x^n$ converges absolutely for $|x| < R$,

then $\sum a_n \left(f(x)\right)^n$ converges absolutely for any continuous function f on $|f(x)| < R$.

Example 1

Since

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \qquad \text{for } |x| < 1$$

Theorem 1 tells us that

$$\frac{1}{1-4x^2} = \sum_{k=0}^{\infty} (4x^2)^k, \qquad \text{for } |4x^2| < 1$$

Example 2

Find the interval of convergence of

$$\sum_{k=0}^{\infty} (e^x - 4)^k$$

and, within this interval, the sum of the series as a function of x.

Example 2 (continued)

Solution: $\sum_{k=0}^{\infty}(e^x - 4)^k$; Using the Ratio Test for Absolute Convergence:

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{\left|(e^x - 4)^{k+1}\right|}{|(e^x - 4)^k|}$$

$$= \lim_{k\to\infty} |e^x - 4|$$

$$= |e^x - 4| \lim_{k\to\infty} 1$$

$$= |e^x - 4|$$

Therefore the series converges absolutely when $\rho = |e^x - 4| < 1$.

Example 2 (continued)

$$\rho = |e^x - 4| < 1$$
$$-1 < e^x - 4 < 1$$
$$3 < e^x < 5$$
$$\ln 3 < x < \ln 5$$

Let’s check what happens to the series at the endpoint of this interval.

Example 2 (continued)

At $x = \ln 3$, the series becomes

$$\sum_{k=0}^{\infty}\left(e^{\ln 3} - 4\right)^k = \sum_{k=0}^{\infty}(3-4)^k = \sum_{k=0}^{\infty}(-1)^k$$

which diverges.

At $x = \ln 5$, the series becomes

$$\sum_{k=0}^{\infty}\left(e^{\ln 5} - 4\right)^k = \sum_{k=0}^{\infty}(5-4)^k = \sum_{k=0}^{\infty} 1$$

which diverges.

Example 2 (continued)

The series $\sum_{k=0}^{\infty}(e^x - 4)^k$ is a convergent geometric series ($a = 1, r = e^x - 4$) when $\ln 3 < x < \ln 5$

and the sum is

$$\frac{a}{1-r} = \frac{1}{1-(e^x - 4)} = \frac{1}{5 - e^x}.$$

Power Series as Functions

If $\sum_{k=0}^{\infty} c_k (x-a)^k$ converges for $|x-a| < R$ (that is, $a - R < x < a + R$), then define

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k, a - R < x < a + R$$

We can find $f'(x)$ and $\int f(x)\, dx$ as follows:

Term by Term Differentiation and Integration

(a) $f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}\left(c_k (x-a)^k\right)$

$= \sum_{k=0}^{\infty} k c_k (x-a)^{k-1}$

(b) $\int f(x)\,dx = \sum_{k=0}^{\infty} \int c_k (x-a)^k\,dx$

$= \sum_{k=0}^{\infty} \frac{c_k}{k+1} (x-a)^{k+1} + C$

Both have radius of convergence R and interval of convergence $|x-a| < R$.

Series Multiplication

If $\sum a_n x^n$ and $\sum b_n x^n$ converge absolutely for $|x| < R$ and

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

then

$$\left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum c_n x^n$$

which also converges for $|x| < R$.

Example 3

The series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

converges to e^x for all x.

(a) Find the series for $\frac{d}{dx}(e^x)$.

(b) Find the series for $\int e^x \, dx$.

(c) Find the series for e^{-x}.

(d) Multiply the series for e^{-x} and e^x to find $e^{-x}e^x$.

Example 3 (continued)

Solution (a): $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$\frac{d}{dx}(e^x) = \sum_{k=0}^{\infty} \frac{d}{dx}\left(\frac{x^k}{k!}\right)$$

$$= \sum_{k=0}^{\infty} k\,\frac{x^{k-1}}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

Example 3 (continued)

Solution (b): $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$\int e^x\,dx = \sum_{k=0}^{\infty} \int \frac{x^k}{k!}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)\cdot k!} + C$$

$$= \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} + C$$

$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + C$$

$$= -1 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + C$$

$$= -1 + \sum_{k=0}^{\infty} \frac{x^k}{k!} + C$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} + C = e^x + C$$

Example 3 (continued)

Solution (c): $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$$

Example 3 (continued)

<u>Solution (d)</u>: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $e^{-x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$

$$e^{-x}e^x = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}\right) = \sum_{n=0}^{\infty} c_n x^n$$

where $a_k = \frac{1}{k!}$ and $b_k = \frac{(-1)^k}{k!}$ and
$c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Example 3 (continued)

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{(-1)^{n-k}}{(n-k)!} = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!\,(n-k)!}$$

$$c_0 = \frac{(-1)^{0-0}}{0!\,(0-0)!} = 1$$

$$c_1 = \frac{(-1)^{1-0}}{0!\,(1-0)!} + \frac{(-1)^{1-1}}{1!\,(1-1)!} = -1 + 1 = 0$$

$$c_2 = \frac{(-1)^{2-0}}{0!\,(2-0)!} + \frac{(-1)^{2-1}}{1!\,(2-1)!} + \frac{(-1)^{2-2}}{2!\,(2-2)!} = \frac{1}{2} - 1 + \frac{1}{2} = 0$$

etc.

$$c_n = 0, \qquad n \neq 0$$

Example 3 (continued)

$$e^{-x}e^{x} = \left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!}\right)$$

$$= \sum_{n=0}^{\infty} c_n x^n$$

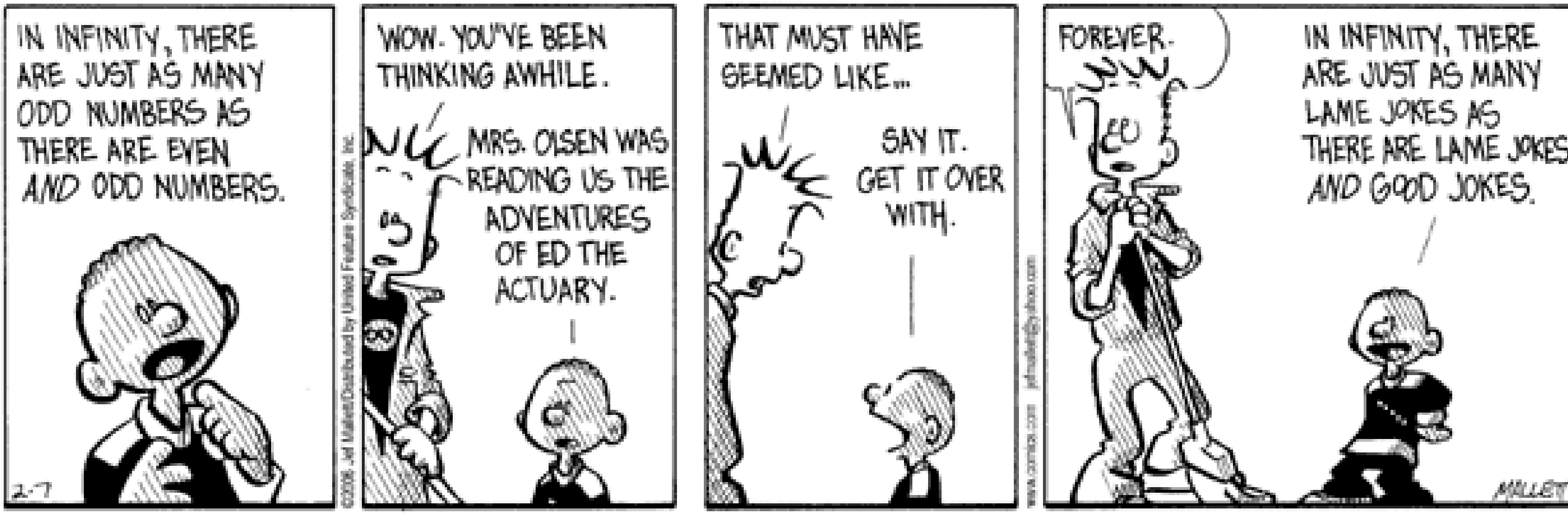
$$= 1 \cdot x^0 + 0 \cdot x^1 + 0 \cdot x^2 + 0 \cdot x^3 + \cdots$$

$$= 1$$

Advice

Read the “Power Series” section in your textbook (including its exercises) –

it will provide you with some excellent examples of how to identify a power series as a function by looking at either the derivative, the antiderivative of the series, or the product to two known series.



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