# Power Series

Part 1

# Power Series

Suppose $x$ is a variable and $c_k$ & $a$ are constants.

A **power series about $\boldsymbol{x = 0}$** is

$$\sum_{k=0}^{\infty} c_k x^k$$

A **power series about $\boldsymbol{x = a}$** is

$$\sum_{k=0}^{\infty} c_k (x - a)^k$$

$a =$ **center** of the power series

$c_k =$ **coefficients** of the power series

# Examples of Power Series

- $\sum_{k=0}^{\infty} x^k$ $\quad a = 0, c_k = 1$
- $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ $\quad a = 0, c_k = \frac{1}{k!}$
- $\sum_{k=0}^{\infty} k!\, x^k$ $\quad a = 0, c_k = k!$
- $\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{3^k(k+1)}$ $\quad a = 0, c_k = \frac{(-1)^k}{3^k(k+1)}$
- $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$ $\quad a = 5, c_k = \frac{1}{k^2}$

# Series That Are Not Power Series

- $\sum_{k=0}^{\infty} \frac{1}{x^k}$
- $\sum_{k=0}^{\infty} \sin x$
- $\sum_{k=0}^{\infty} e^{-x}$
- $\sum_{k=0}^{\infty} \ln x$

# Question

For what values of $x$ does a given power series converge?

# Power Series Convergence Theorem

For a power series $\sum c_k(x-a)^k$, exactly one of the following is true:

(a) The series converges only for $x = a$.

(b) The series converges absolutely for all $x$.

(c) The series converges absolutely for all $x$ in some finite open interval $(a-R, a+R)$ and diverges if $x < a-R$ or $x > a+R$.

At the points $x = a-R$ and $x = a+R$, the series may converge (absolutely or conditionally) or diverge.

$R =$ **radius of convergence**

$(a-R, a+R) =$ **interval of convergence**

# Example 1

Find the interval of convergence and radius of convergence for

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots$$

# Example 1 (continued)

Solution: $\sum_{k=0}^{\infty} x^k$

$\sum_{k=0}^{\infty} x^k$ is a geometric series with $a = 1$ and $r = x$.

Therefore, the series
- converges absolutely for $|x| < 1$
- diverges for $|x| \geq 1$.

So the
- interval of convergence is $(-1,1)$
- radius of convergence is $R = 1$ (half the width of the interval of convergence).

# Example 1 (continued)

Also, note that since $\sum_{k=0}^{\infty} x^k$ is a geometric series with $a = 1$ and $r = x$,

$$\sum_{k=0}^{\infty} x^k = \frac{a}{1-r} = \frac{1}{1-x}, \qquad \text{for } |x| < 1$$

# Example 2

Find the interval of convergence and radius of convergence for

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots$$

Note: 0! = 1

# Example 2 (continued)

Solution: $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

Using the Ratio Test for Absolute Convergence:

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{\left|\frac{x^{k+1}}{(k+1)!}\right|}{\left|\frac{x^k}{k!}\right|}$$

$$= \lim_{k\to\infty} \left|\frac{x^{k+1}}{(k+1)!}\right| \cdot \left|\frac{k!}{x^k}\right| = \lim_{k\to\infty} \frac{|x|}{(k+1)} = 0 < 1$$

# Example 2 (continued)

Therefore the series converges absolutely for any $x$.

So the

- interval of convergence is $(-\infty, \infty)$
- radius of convergence is $R = \infty$.

# Important Limit

Example 2 tells us that $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all $x$, therefore:

$$\lim_{k \to \infty} \frac{x^k}{k!} = 0$$

# Example 3

Find the interval of convergence and radius of convergence for

$$\sum_{k=0}^{\infty} k!\, x^k$$

# Example 3 (continued)

Solution: $\sum_{k=0}^{\infty} k!\, x^k$

If $x = 0$, then $\sum_{k=0}^{\infty} k!\, x^k = \sum_{k=0}^{\infty} k! \cdot 0^k = 0.$

If $x \neq 0$, then the Ratio Test for Absolute Convergence gives:

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{\left|(k+1)!\, x^{k+1}\right|}{|k!\, x^k|}$$

$$= \lim_{k\to\infty} (k+1)|x| = \infty$$

# Example 3 (continued)

Therefore the series diverges for $x \neq 0$.

So the

- interval of convergence is $\{0\}$
- radius of convergence is $R = 0$.

# Example 4

Find the interval of convergence and radius of convergence for

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{3^k(k+1)}$$

$$= 1 - \frac{x}{3(2)} + \frac{x^2}{3^2(3)} - \frac{x^3}{3^3(4)} + \cdots$$

# Example 4 (continued)

Solution: $\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{3^k(k+1)}$; Using the Ratio Test for Absolute Convergence:

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{\left|\frac{x^{k+1}}{3^{k+1}\left((k+1)+1\right)}\right|}{\left|\frac{x^k}{3^k(k+1)}\right|}$$

$$= \lim_{k\to\infty} \left|\frac{x^{k+1}}{3^{k+1}\left((k+1)+1\right)}\right| \cdot \left|\frac{3^k(k+1)}{x^k}\right|$$

$$= \lim_{k\to\infty} \left|\frac{x(k+1)}{3(k+2)}\right| = \frac{|x|}{3} \lim_{k\to\infty} \frac{k+1}{k+2} = \frac{|x|}{3}$$

# Example 4 (continued)

There are three cases we need to consider:

<u>Case 1</u>: $\rho < 1$, in which case the series converges absolutely.

<u>Case 2</u>: $\rho > 1$, in which case the series diverges.

<u>Case 3</u>: $\rho = 1$, in which case we need to look at each case individually in order to determine if it converges (absolutely or conditionally) or if it diverges.

# Example 4 (continued)

Case 1: $\rho = \frac{|x|}{3} < 1 \Rightarrow |x| < 3$, so the series converges absolutely for $|x| < 3$

Case 2: $\rho = \frac{|x|}{3} > 1 \Rightarrow |x| > 3$, so the series diverges for $|x| > 3$

Case 3: $\rho = \frac{|x|}{3} = 1 \Rightarrow |x| = 3 \Rightarrow x = \pm 3$

# Example 4 (continued)

When $x = 3$, the series becomes

$$\sum_{k=0}^{\infty} \frac{(-1)^k 3^k}{3^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)}$$

which is the conditionally convergent alternating harmonic series.

When $x = -3$, the series becomes

$$\sum_{k=0}^{\infty} \frac{(-1)^k (-3)^k}{3^k(k+1)} = \sum_{k=0}^{\infty} \frac{3^k}{3^k(k+1)} = \sum_{k=0}^{\infty} \frac{1}{(k+1)}$$

which is the divergent harmonic series.

# Example 4 (continued)

Therefore the series

- converges absolutely for $|x| < 3$
- converges conditionally for $x = 3$
- diverges for $x \leq -3 \text{ or } 3 < x$

So the

- interval of convergence is $(-3, 3]$
- radius of convergence is $R = 3$ (half the width of the interval of convergence).

# Example 5

Find the interval of convergence and radius of convergence for

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

This is a power series of the form $\sum_{k=0}^{\infty} c_k x^k$ with $c_k = \frac{(-1)^{k/2}}{k!}$ for $k$ even and $c_k = 0$ for $k$ odd.

# Example 5 (continued)

<u>Solution</u>: $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$; Using the Ratio Test for Absolute Convergence:

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{\left|\frac{x^{2(k+1)}}{\left(2(k+1)\right)!}\right|}{\left|\frac{x^{2k}}{(2k)!}\right|}$$

$$= \lim_{k\to\infty} \left|\frac{x^{2(k+1)}}{\left(2(k+1)\right)!}\right| \cdot \left|\frac{(2k)!}{x^{2k}}\right|$$

$$= \lim_{k\to\infty} \left|\frac{x^{2k+2}}{(2k+2)!}\right| \cdot \left|\frac{(2k)!}{x^{2k}}\right|$$

# Example 5 (continued)

$$\lim_{k\to\infty}\left|\frac{x^{2k+2}}{(2k+2)!}\right|\cdot\left|\frac{(2k)!}{x^{2k}}\right|$$

$$=\lim_{k\to\infty}\left|\frac{x^{2k+2}}{(2k+2)\cdot(2k+1)\cdot(2k)!}\cdot\frac{(2k)!}{x^{2k}}\right|$$

$$=\lim_{k\to\infty}\left|\frac{x^{2}}{(2k+2)(2k+1)}\right|$$

$$=|x^2|\lim_{k\to\infty}\frac{1}{(2k+2)(2k+1)}$$

$$=|x^2|\cdot 0=0<1$$

# Example 5 (continued)

Therefore the series converges absolutely for any $x$.

So the

- interval of convergence is $(-\infty, \infty)$
- radius of convergence is $R = \infty$.

# Example 6

Find the interval of convergence and radius of convergence for

$$\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$$

# Example 6 (continued)

Solution: $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$; Using the Ratio Test for Absolute Convergence:

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{\left|\frac{(x-5)^{k+1}}{(k+1)^2}\right|}{\left|\frac{(x-5)^k}{k^2}\right|}$$

$$= \lim_{k\to\infty} \left|\frac{(x-5)^{k+1}}{(k+1)^2}\right| \cdot \left|\frac{k^2}{(x-5)^k}\right|$$

$$= \lim_{k\to\infty} \left|\frac{(x-5)k^2}{(k+1)^2}\right| = |x-5| \lim_{k\to\infty} \frac{k^2}{(k+1)^2}$$

$$= |x-5| \cdot 1 = |x-5|$$

# Example 6 (continued)

Case 1:

$$\rho = |x - 5| < 1$$
$$-1 < x - 5 < 1$$
$$4 < x < 6,$$

so the series converges absolutely for $4 < x < 6$.

Case 2:

$$\rho = |x - 5| > 1$$
$$x - 5 < -1 \text{ or } 1 < x - 5$$
$$x < 4 \text{ or } 6 < x,$$

so the series diverges for $x < 4$ or $6 < x$.

Case 3:

$$\rho = |x - 5| = 1$$
$$x - 5 = \pm 1$$
$$x = 6 \text{ or } x = 4.$$

# Example 6 (continued)

When $x = 6$, the series becomes

$$\sum_{k=1}^{\infty} \frac{(6-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(1)^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which is a convergent $p$-series ($p = 2$).

When $x = 4$, the series becomes

$$\sum_{k=1}^{\infty} \frac{(4-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

Since

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

is a convergent $p$-series ($p = 2$), the series converges absolutely at $x = 4$.

# Example 6 (continued)

Therefore the series

- converges absolutely for $[4,6]$
- diverges for $x < 4 \text{ or } 6 < x$

So the

- interval of convergence is $[4,6]$
- radius of convergence is $R = 1$ (half the width of the interval of convergence).


http://math.sfsu.edu/beck/images/calvin.hobbes.bushel.gif