Alternating Series, Absolute and Conditional Convergence

Part 2

Absolute and Conditional Convergence

Absolute Convergence

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

converges absolutely if

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + |u_3| + \cdots + |u_k| + \cdots$$

converges.

Theorem

If a series converges absolutely, then it converges.

Example 1

Does the series

$$1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$$

converge or diverge?

<u>Solution</u>:

Since

$$|1| + \left|-\frac{1}{2}\right| + \left|-\frac{1}{2^2}\right| + \left|\frac{1}{2^3}\right| + \left|\frac{1}{2^4}\right| + \left|-\frac{1}{2^5}\right| + \left|-\frac{1}{2^6}\right| + \cdots = \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$$

is a convergent geometric series ($a = 1, r = 1/2$), the original series converges.

Example 2

Show that the series

$$\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$$

converges.

Example 2 (continued)

Solution: $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$

Notice that

$$\left|\frac{\cos k}{k^2}\right| \leq \frac{1}{k^2}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series ($p = 2$), the series $\sum_{k=1}^{\infty} \left|\frac{\cos k}{k^2}\right|$ converges by the Comparison Test.

Therefore, the series $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ converges.

Conditionally Convergent

Consider the alternating harmonic series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1}\frac{1}{k} + \cdots$$

It is convergent, but it is not absolutely convergent.

Such series are called **conditionally convergent.**

The Ratio Test for Absolute Convergence

Let $\sum u_k$ be a series with nonzero terms and suppose

$$\lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \rho$$

a) If $\rho < 1$, the series converges absolutely.

b) If $\rho > 1$ or $\rho = \infty$, the series diverges.

c) If $\rho = 1$, the series may converge or diverge.

Example 3

The alternating series

$$\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}$$

converges absolutely since

$$\rho = \lim_{k\to\infty} \frac{\left(\frac{2^{k+1}}{(k+1)!}\right)}{\left(\frac{2^k}{k!}\right)}$$

$$= \lim_{k\to\infty} \frac{2}{k+1} = 0 < 1$$

Rearranging Terms

IMPORTANT:

You cannot always rearrange the order of the terms of a series and get the same sum!

(Very weird, but true!)

Theorem

If $\sum a_n$ converges absolutely and

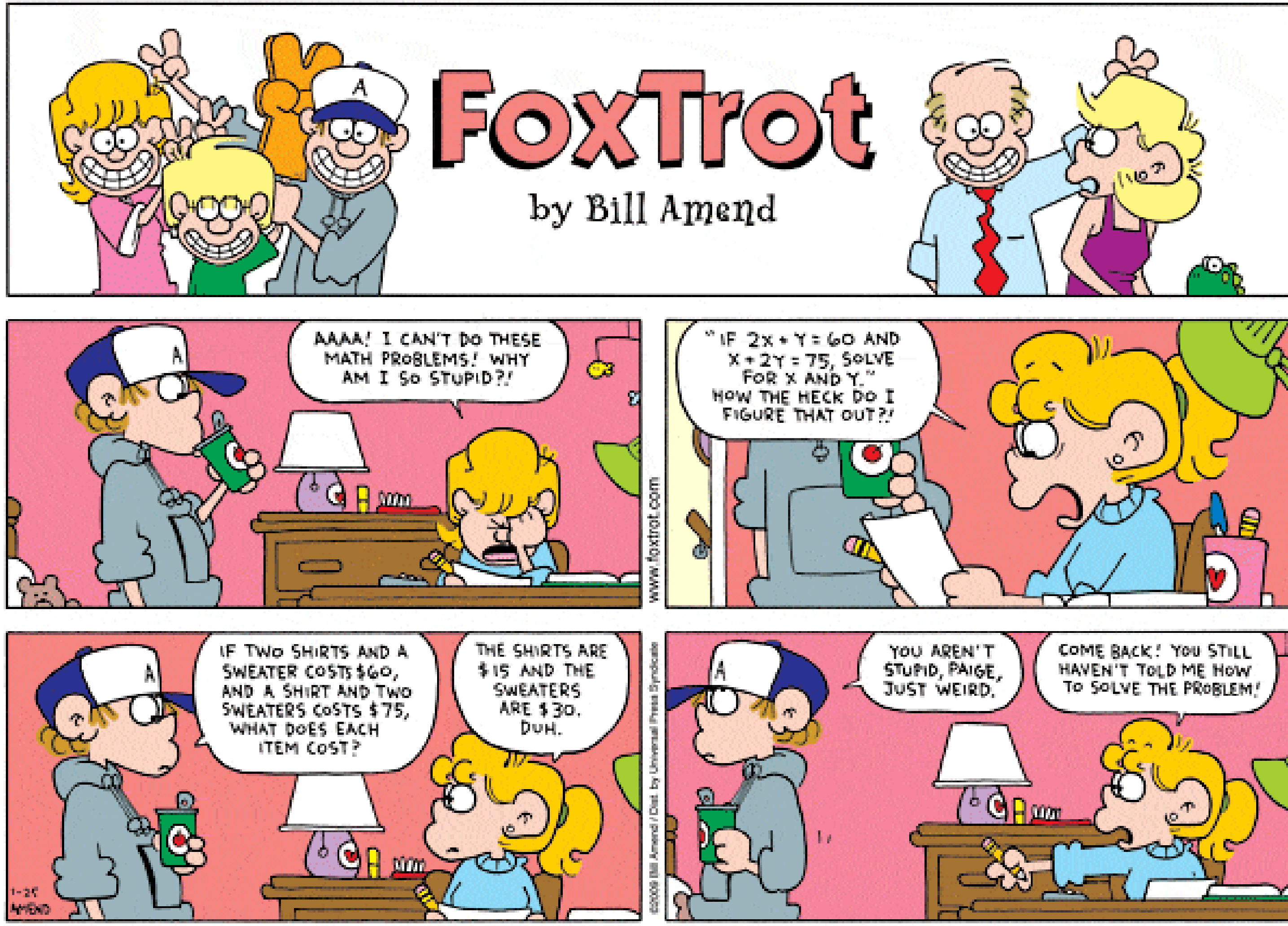
$$b_1, b_2, b_3, \dots, b_k, \dots$$

is any rearrangement of

$$a_1, a_2, a_3, \dots, a_k, \dots$$

then

$$\sum a_n = \sum b_n .$$



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