Infinite Series

Part 3

Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

The **harmonic series** diverges.

Proof:

We will show that the sequence of partial sums, $\{s_n\}$, is unbounded.

Harmonic Series (continued)

$$s_1 = 1$$

$$s_2 = 1 + \frac{1}{2}$$

$$s_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$s_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

Clearly

$$s_1 < s_2 < s_3 < \cdots < s_4 < \cdots$$

Harmonic Series (continued)

$$s_2 = 1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = \frac{2}{2}$$

$$s_{2^2} = s_2 + \frac{1}{3} + \frac{1}{4} > s_2 + \frac{1}{4} + \frac{1}{4} > \frac{3}{2}$$

$$s_{2^3} = s_{2^2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > s_{2^2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} > \frac{4}{2}$$

If we continue on we see

$$s_{2^n} > \frac{n+1}{2}$$

Harmonic Series (continued)

Since for any M we can find an n such that

$$\frac{n+1}{2} > M,$$

the sequence is unbounded.

Hence the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges.

Theorem 1

If $\sum u_k$ converges, then

$$\lim_{k\to\infty} u_k = 0.$$

<u>IMPORTANT</u>:

If $\lim_{k\to\infty} u_k = 0$, then we do not know if the series converges or diverges!

The n-th Term Test for Divergence

If $\lim_{k \to \infty} u_k \neq 0$ then the series diverges.

Example 2

The series

$$\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$$

diverges since

$$\lim_{k\to\infty} \frac{k}{k+1} = 1 \neq 0.$$

Combining Series

If $\sum u_k$ and $\sum v_k$ are convergent series then $\sum(u_k + v_k)$ and $\sum(u_k - v_k)$ are convergent series and the sums of these series are related by

$$\sum(u_k + v_k) = \sum u_k + \sum v_k$$

$$\sum(u_k - v_k) = \sum u_k - \sum v_k$$

Combining Series (continued)

If c is a nonzero constant, then the series $\sum u_k$ and $\sum cu_k$ both converge or both diverge.

In the case of convergence, the sums are related by

$$\sum cu_k = c \sum u_k$$

Combining Series (continued)

Convergence or divergence is unaffected by deleting a finite number of terms from the beginning of the series; that is, for any positive integer K, the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots$$

and

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \cdots$$

both converge or both diverge.

Example 3

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k}-\frac{2}{5^{k-1}}\right)$$

$$=\sum_{k=1}^{\infty}\frac{3}{4}\left(\frac{1}{4}\right)^{k-1}-\sum_{k=1}^{\infty}2\left(\frac{1}{5}\right)^{k-1}$$

$$=\frac{3/4}{1-1/4}-\frac{2}{1-1/5}$$

$$=1-\frac{10}{4}=-\frac{3}{2}$$

Example 4

$$\sum_{k=1}^{\infty} \frac{5}{k}$$

$$= \sum_{k=1}^{\infty} 5\left(\frac{1}{k}\right)$$

$$= 5 \sum_{k=1}^{\infty} \frac{1}{k}$$

Therefore $\sum_{k=1}^{\infty} \frac{5}{k}$ diverges since $\sum_{k=1}^{\infty} \frac{1}{k}$ (the harmonic series) diverges.

Example 5

$$\sum_{k=10000}^{\infty} \frac{1}{k}$$

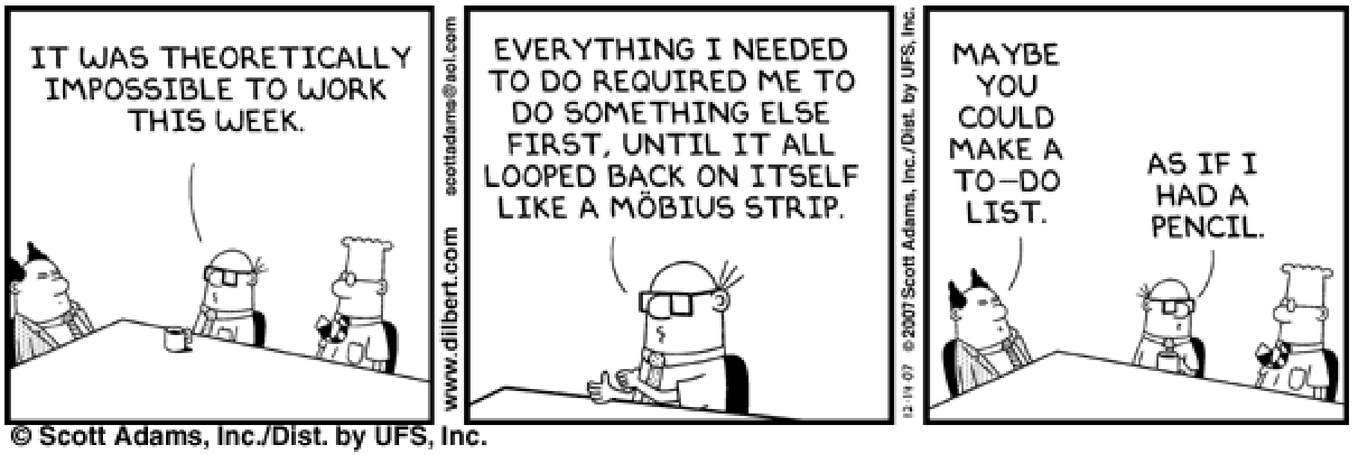
diverges since

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges.

Theorem 2

If $\sum u_k$ is a series with positive terms and if $\{s_n\}$ is bounded then the series converges. Otherwise, the series diverges.



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