Infinite Series

Part 2:

Geometric Series

Geometric Series

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + \cdots + ar^{k-1} + \cdots$$

$$(a \neq 0)$$

r is the **ratio** for the series

Example 1

The following are geometric series:

- $1 + 2 + 4 + 8 + \cdots + 2^{k-1} + \cdots$
  - $(a = 1, r = 2)$
- $3 + \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^{k-1}} + \cdots$
  - $\left(a = 3, r = \frac{1}{10}\right)$
- $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots + (-1)^{k+1}\frac{1}{2^k} + \cdots$
  - $\left(a = \frac{1}{2}, r = -\frac{1}{2}\right)$
- $1 - 1 + 1 - 1 + \cdots + (-1)^{k+1} + \cdots$
  - $(a = 1, r = -1)$

Theorem

A geometric series

$$a + ar + ar^2 + \cdots + ar^{k-1} + \cdots$$

converges if $|r| < 1$ and diverges if $|r| \geq 1$.

When the series converges, the sum is

$$\frac{a}{1-r} = a + ar + ar^2 + \cdots + ar^{k-1} + \cdots$$

Proof of the Theorem

If $r = 1$, it clearly diverges. So suppose $r \neq 1$.

$$s_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$$
$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

$$s_n - rs_n = a - ar^n$$
$$(1 - r)s_n = a - ar^n$$

Therefore, since $r \neq 1$ we have:

$$s_n = \frac{a - ar^n}{1 - r}$$

Proof of the Theorem (continued)

$$s_n = \frac{a - ar^n}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right)$$

$$= \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n$$

And we have:

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } |r| < 1 \\ \text{dne}, & \text{if } |r| > 1 \end{cases}$$

Proof of the Theorem (continued)

That is

$$\lim_{n\to\infty} s_n = \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n$$

$$= \begin{cases} \dfrac{a}{1-r}, & \text{if } |r| < 1 \\ \text{dne}, & \text{if } |r| > 1 \end{cases}$$

proving the theorem.

Example 2

Does the series

$$5 + \frac{5}{4} + \frac{5}{4^2} + \cdots + \frac{5}{4^{k-1}} + \cdots$$

converge or diverge? If it converges, what is its sum?

Example 2 (continued)

Solution:

$$5 + \frac{5}{4} + \frac{5}{4^2} + \cdots + \frac{5}{4^{k-1}} + \cdots$$

$$= \sum_{k=1}^{\infty} 5\left(\frac{1}{4}\right)^{k-1}$$

So this is a geometric series with $a = 5$ and $r = \frac{1}{4} < 1.$

The series converges and has sum $\frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}.$

Example 3

Find the rational number represented by

0.784784784…

<u>Solution</u>:

$$0.784784784\ldots = 0.784 + 0.000784 + 0.000000784 + \cdots$$

$$= \frac{0.784}{1} + \frac{0.784}{1000} + \frac{0.784}{1000^2} + \cdots$$

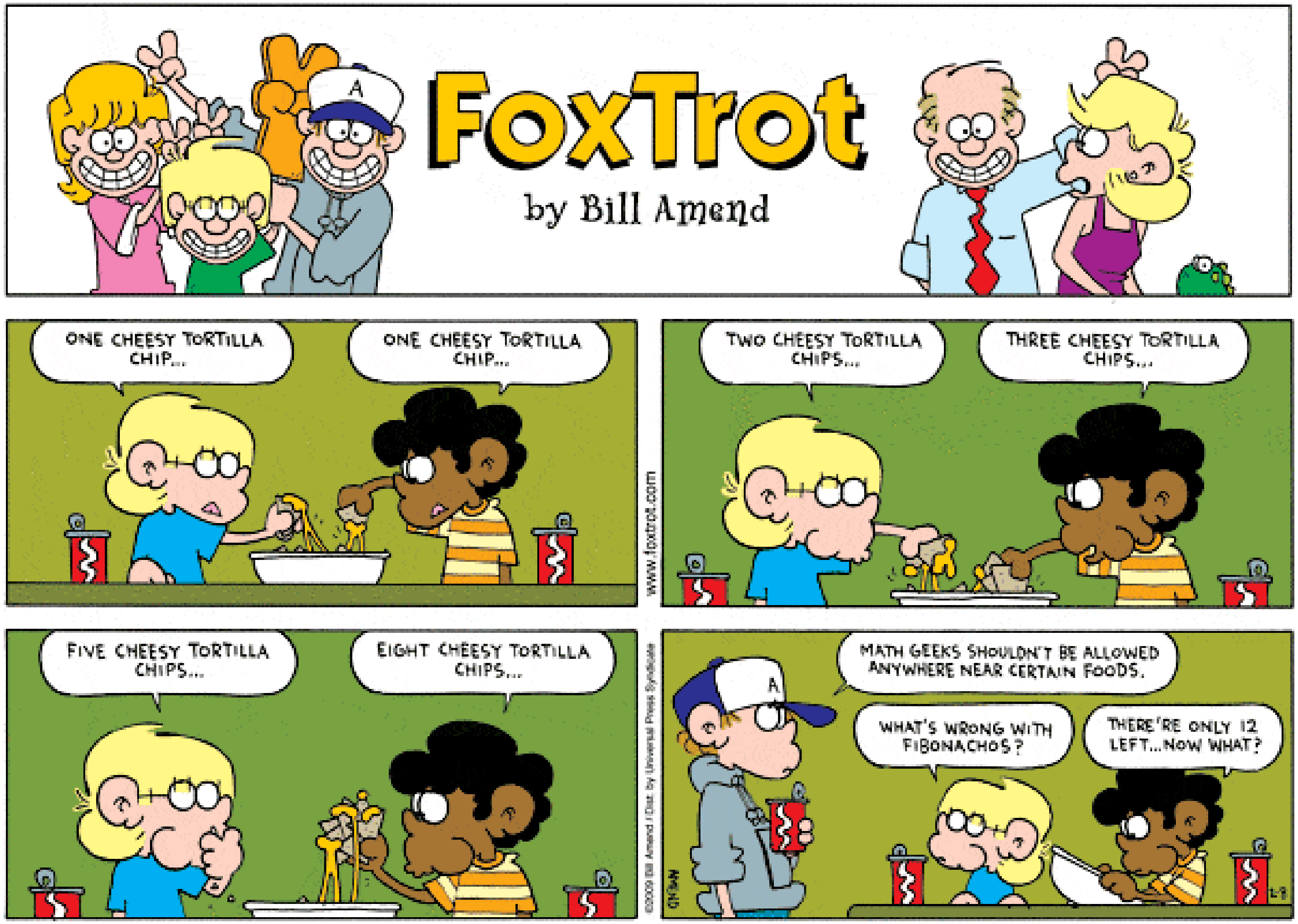
$$= \sum_{k=1}^{\infty} 0.784 \left(\frac{1}{1000}\right)^{k-1}$$

Example 3 (continued)

$$\sum_{k=1}^{\infty} 0.784\left(\frac{1}{1000}\right)^{k-1}$$

This is a geometric series with $a = 0.784$ and $r = \frac{1}{1000} < 1.$

The series converges and has sum $\frac{a}{1-r} = \frac{0.784}{1-\frac{1}{1000}} = \boxed{\frac{784}{999}}.$



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