Sequences

Part 2

Sequences

To write a sequence without specifying the numerical values:

$$a_1, a_2, \dots, a_n, \dots = \{a_n\}_{n=1}^{+\infty} = \{a_n\}$$

We can think of a sequence as a function f whose domain is the set of positive integers and

$$f(n) = a_n.$$

Now we can look at the graph of the sequence.

Example 1

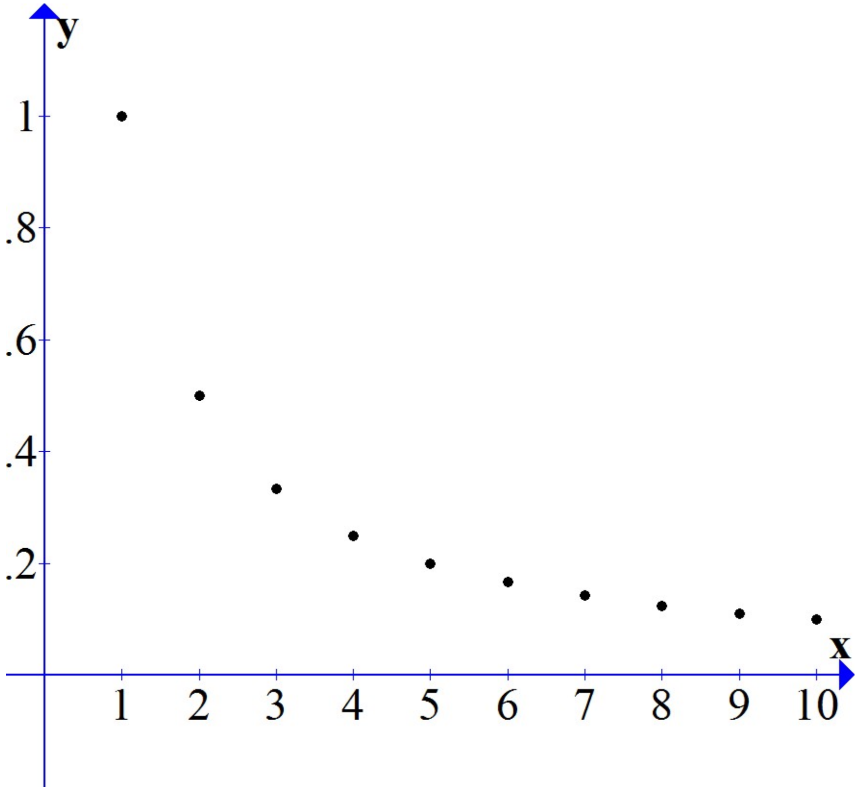
The graph of

$$\left\{\frac{1}{n}\right\}_{n=1}^{+\infty}$$

is the graph of

$$y = f(x) = \frac{1}{x},$$

$$x = 1, 2, 3, \ldots$$



Convergence and Divergence

If a sequence $\{a_n\}_{n=1}^{+\infty}$ has a limit L (that is, the terms in the sequence become arbitrarily close to the number L as n gets larger), we say that the sequence **converges** to L and write

$$\lim_{n\to\infty} a_n = L$$

A sequence that does not have a finite limit is said to **diverge**.

Properties of Limits for Sequences

Suppose $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$ and that c is a constant. Then:

a) $\lim_{n\to\infty} c = c$

b) $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n = cL_1$

c) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L_1 + L_2$

d) $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L_1 - L_2$

e) $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = L_1 L_2$

f) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L_1}{L_2} \quad (L_2 \neq 0)$

Example 2

Do the following sequences converge or diverge? If it converges, find the limit.

a) $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$

b) $\left\{(-1)^{n+1}\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$

c) $\left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{+\infty}$

d) $\{8-2n\}_{n=1}^{+\infty}$

e) $\left\{\frac{n}{e^n}\right\}_{n=1}^{+\infty}$

f) $\{\sqrt[n]{n}\}_{n=1}^{+\infty}$

Example 2 (continued)

Solution a): $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$

$\frac{n}{2n+1}$ is a rational function of n. Since the degree of the numerator is equal to the degree of the denominator, the limit as n goes to ∞ is equal to the ratio of the leading coefficients.

$$\lim_{n\to+\infty}\frac{n}{2n+1}=\frac{1}{2}$$

So $\boxed{\text{the sequence converges and its limit is } \frac{1}{2}.}$

Example 2 (continued)

<u>Solution b)</u>: $\left\{(-1)^{n+1}\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$

$$\lim_{n\to+\infty}(-1)^{n+1}\frac{n}{2n+1}$$

$$=\lim_{n\to+\infty}(-1)^{n+1}\cdot\lim_{n\to+\infty}\frac{n}{2n+1}$$

$$=\lim_{n\to+\infty}(-1)^{n+1}\cdot\frac{1}{2}=\frac{1}{2}\lim_{n\to+\infty}(-1)^{n+1}$$

and this limit does not exist because it alternates between -1 and 1.

Therefore, the sequence diverges.

Example 2 (continued)

Solution c): $\left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{+\infty}$

$$\lim_{n\to+\infty}(-1)^{n+1}\frac{1}{n}$$

$$=\lim_{n\to+\infty}(-1)^{n+1}\cdot\lim_{n\to+\infty}\frac{1}{n}$$

$$=\lim_{n\to+\infty}(-1)^{n+1}\cdot 0=0$$

Therefore, the sequence converges and its limit is 0.

Example 2 (continued)

Solution d): $\{8-2n\}_{n=1}^{+\infty}$

$$\lim_{n\to+\infty}(8-2n)=-\infty$$

So $\boxed{\text{the sequence diverges.}}$

Solution e): $\left\{\frac{n}{e^n}\right\}_{n=1}^{+\infty}$

$$\lim_{n\to+\infty}\frac{n}{e^n}\overset{\text{L'Hop}}{=}\lim_{n\to+\infty}\frac{1}{e^n}=0$$

So $\boxed{\text{the sequence converges and the limit is 0.}}$

Example 2 (continued)

Solution f): $\{\sqrt[n]{n}\}_{n=1}^{+\infty}$

$$\lim_{n\to+\infty} \sqrt[n]{n} = \lim_{n\to+\infty} n^{1/n}$$
$$= \lim_{n\to+\infty} e^{\ln n^{1/n}}$$
$$= \lim_{n\to+\infty} e^{\left(\frac{1}{n}\ln n\right)}$$
$$= e^{\left[\lim_{n\to+\infty}\left(\frac{1}{n}\ln n\right)\right]}$$

Example 2 (continued)

Now,

$$\lim_{n\to+\infty}\left(\frac{1}{n}\ln n\right) = \lim_{n\to+\infty}\left(\frac{\ln n}{n}\right)$$

$$\overset{\text{L'Hop}}{=} \lim_{n\to+\infty}\left(\frac{1/n}{1}\right)$$

$$= \lim_{n\to+\infty}\frac{1}{n} = 0$$

Example 2 (continued)

Using this we get:

$$e^{\left[\lim_{n\to+\infty}\left(\frac{1}{n}\ln n\right)\right]} = e^0 = 1$$

That is,

$$\lim_{n\to+\infty} \sqrt[n]{n} = 1$$

So $\boxed{\{\sqrt[n]{n}\}_{n=1}^{+\infty} \text{ converges and its limit is 1.}}$

Theorem

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}_{n=1}^{+\infty}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} f(a_n) = L$$

Helpful Limits

- $\lim_{x\to+\infty} \frac{\sin x}{x} = 0$ and $\lim_{x\to+\infty} \frac{\cos x}{x} = 0$

- If $p(x)$ and $q(x)$ are polynomials with no factors in common and if the degree of $p(x)$ is n and the degree of q(x) is m, then:
  - If $n > m$ then $\lim_{x\to+\infty} \frac{p(x)}{q(x)} = +\infty$ or $\lim_{x\to+\infty} \frac{p(x)}{q(x)} = -\infty$.
  - If $n < m$ then $\lim_{x\to+\infty} \frac{p(x)}{q(x)} = 0$.
  - If $n = m$ then $\lim_{x\to+\infty} \frac{p(x)}{q(x)} = \frac{\text{leading coefficient of } p\,(x)}{\text{leading coefficient of } q\,(x)}$.

Six Convergent Sequences

- $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
- $\lim_{n\to\infty} x^{1/n} = 1 \ \ (x > 0)$
- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim_{n\to\infty} x^n = 0 \ \ (|x| < 1)$
- $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

L'Hôpital Controversy

From: Wikipedia – John Bernoulli

Bernoulli was hired by Guillaume de L'Hôpital for tutoring in mathematics. Bernoulli and L'Hôpital signed a contract which gave L'Hôpital the right to use Bernoulli's discoveries as he pleased. L'Hôpital authored the first textbook on infinitesimal calculus, *Analyse des Infiniment Petits pour l'Intelligence de Lignes Courbes* in 1696, which mainly consisted of the work of Bernoulli, including what is now known as L'Hôpital's rule.

Subsequently, in letters to Leibniz, Varignon and others, Bernoulli complained that he had not received enough credit for his contributions, in spite of the fact that L'Hôpital acknowledged fully his debt in the preface of his book:

…

"I recognize I owe much to Messrs. Bernoulli's insights, above all to the young (John), currently a professor in Groningue. I did unceremoniously use their discoveries, as well as those of Mr. Leibniz. For this reason I consent that they claim as much credit as they please, and will content myself with what they will agree to leave me."