Improper Integrals

Part 3:

Tests for Convergence and Divergence

Useful Fact #1

First, if $n > 1$ and $a > 0$, then

$$\int_a^\infty \frac{1}{x^n}\,dx = \lim_{l\to\infty}\int_a^l \frac{1}{x^n}\,dx$$

$$= \lim_{l\to\infty}\left(\frac{1}{-n+1}x^{-n+1}\bigg|_a^l\right) = \frac{1}{-n+1}\lim_{l\to\infty}\left(\frac{1}{x^{n-1}}\bigg|_a^l\right)$$

$$= \frac{1}{-n+1}\lim_{l\to\infty}\left(\frac{1}{l^{n-1}} - \frac{1}{a^{n-1}}\right)$$

$$= \frac{1}{-n+1}\left(0 - \frac{1}{a^{n-1}}\right) = \frac{1}{a^{n-1}(n-1)}$$

$\int_a^\infty \frac{1}{x^n}\,dx$ converges for $n > 1$ and $a > 0$

Useful Fact #2

First, if $n < 1$ and $a > 0$, then

$$\int_a^\infty \frac{1}{x^n}\,dx = \lim_{l\to\infty} \int_a^l \frac{1}{x^n}\,dx$$

$$= \lim_{l\to\infty}\left(\frac{1}{1-n}x^{1-n}\Big|_a^l\right)$$

$$= \frac{1}{1-n}\lim_{l\to\infty}\left(l^{1-n} - a^{1-n}\right)$$

$$= \infty$$

$\int_a^\infty \frac{1}{x^n}\,dx$ diverges for $n < 1$ and $a > 0$

Useful Fact #3

First, if $n = 1$ and $a > 0$, then

$$\int_a^\infty \frac{1}{x}dx = \lim_{l\to\infty}\int_a^l \frac{1}{x}dx$$

$$= \lim_{l\to\infty}\left(\ln x\Big|_a^l\right)$$

$$= \lim_{l\to\infty}(\ln l - \ln a) = \infty$$

$\int_a^\infty \frac{1}{x}dx$ diverges for $a > 0$

Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.
2. $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \text{ and } \int_a^\infty g(x)\,dx$$

both converge or both diverge.

Example 1

Perform a test to determine if

$$\int_3^\infty \frac{1}{x^3+7}\,dx$$

converges or diverges.

<u>Solution</u>:

We need to compare $\frac{1}{x^3+7}$ to a function $g(x)$ for which we know whether $\int_3^\infty g(x)\,dx$ converges or diverges.

Example 1 (continued)

If we let $g(x) = \frac{1}{x^3}$ then, by the Useful Fact #1, $\int_3^\infty g(x)\,dx$ converges.

Note that

$$0 \leq \frac{1}{x^3 + 7} \leq \frac{1}{x^3}.$$

So $\int_3^\infty \frac{1}{x^3+7}\,dx$ converges by the Direct Comparison Test with $\int_3^\infty \frac{1}{x^3}\,dx$.

Example 2

Perform a test to determine if

$$\int_2^\infty \frac{1}{\sqrt{x^2-1}}\,dx$$

converges or diverges.

<u>Solution</u>:

We need to compare $\frac{1}{\sqrt{x^2-1}}$ to a function $g(x)$ for which we know whether $\int_2^\infty g(x)\,dx$ converges or diverges.

Example 2 (continued)

If we let $g(x) = \frac{1}{x}$ then, by the Useful Fact #3, $\int_2^\infty g(x)\,dx$ diverges.

Since

$$g(x) = \frac{1}{x} < \frac{1}{\sqrt{x^2 - 1}}$$

We cannot use the Direct Comparison test which requires $0 \leq f(x) \leq g(x)$. Let's try the Limit Comparison Test.

Example 2 (continued)

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty}\frac{x}{\sqrt{x^2-1}}$$

This is an indeterminate form of type $\frac{\infty}{\infty}$. We can use L'Hopital's Rule to evaluate this limit.

Another way is to re-write the limit as follows:

Example 2 (continued)

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2-1}}=\lim_{x\to\infty}\frac{x}{\sqrt{x^2-1}}\cdot\frac{1/x}{1/\sqrt{x^2}}$$

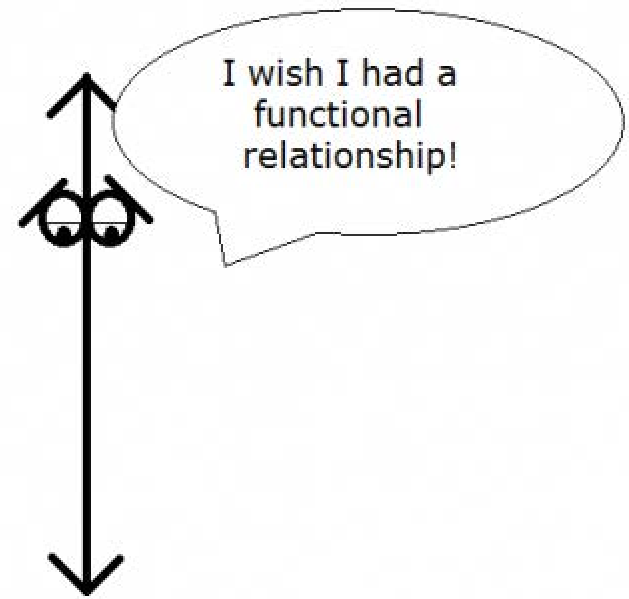
$$=\lim_{x\to\infty}\frac{1}{\sqrt{1-1/x^2}}$$

$$=\frac{1}{\sqrt{1-0}}=1$$

Example 2 (continued)

Since $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ is positive and finite,

$\int_2^\infty \frac{1}{\sqrt{x^2-1}}\,dx$ diverges by the Limit Comparison Test with $\int_2^\infty \frac{1}{x}\,dx$.



http://math-fail.com/page/16