Improper Integrals

Part 2:

Integrals Which Are Not Bounded on the Interval

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Let f be continuous on $[a, b)$ and suppose that $\lim_{x\to b^-} f(x) = -\infty$ or that $\lim_{x\to b^-} f(x) = +\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{l\to b^-} \int_a^l f(x)\,dx$$

- If this limit exists, the improper integral is said to **converge**, and the value of the limit is the value of the integral.
- If this limit does not exist, then the improper integral **diverges**, and the integral has no value.

Integrals Which Are Not Bounded on the Interval (continued)

Let f be continuous on $(a, b]$ and suppose that $\lim_{x\to a^+} f(x) = -\infty$ or that $\lim_{x\to a^+} f(x) = +\infty$. Then

$$\int_a^b f(x)\,dx = \lim_{l\to a^+} \int_l^b f(x)\,dx$$

- If this limit exists, the improper integral is said to **converge**, and the value of the limit is the value of the integral.
- If this limit does not exist, then the improper integral **diverges**, and the integral has no value.

Integrals Which Are Not Bounded on the Interval (continued)

Let f be continuous on $[a, b]$ with the exception that at some point c satisfying $a < c < b$, $f(x)$ becomes infinite as x approaches c from the left or the right. Then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

- If both integrals on the right exist, then $\int_a^b f(x)\,dx$ is said to **converge**.
- If either integral on the right does not exist, then $\int_a^b f(x)\,dx$ **diverges**.

Example 1

Does the following improper integral converge or diverge? If it converges, evaluate the integral.

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$$

Example 1 (continued)

<u>Solution</u>:

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{l\to 1^-} \int_0^l \frac{1}{\sqrt{1-x}}\,dx$$

$$= \lim_{l\to 1^-} \left[-2\sqrt{1-x}\Big|_0^l \right] = \lim_{l\to 1^-} \left[-2\sqrt{1-l} + 2\sqrt{1-0} \right]$$

$$= 0 + 2 = 2$$

Therefore $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$ converges and $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = 2.$

Example 2

Does the following improper integral converge or diverge? If it converges, evaluate the integral.

$$\int_{1}^{2} \frac{1}{1-x}\,dx$$

Example 2 (continued)

Solution:

$$\int_1^2 \frac{1}{1-x}\,dx = \lim_{l\to 1^+} \int_l^2 \frac{1}{1-x}\,dx$$

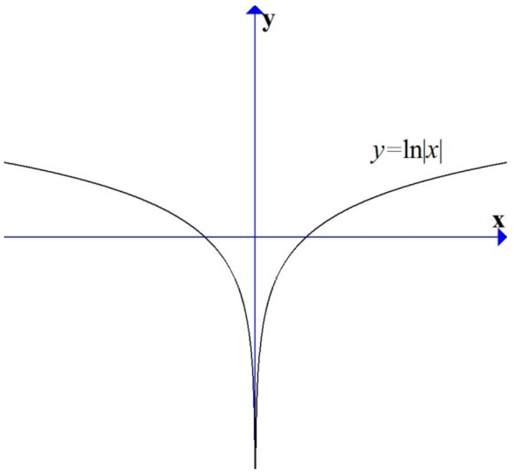
$$= \lim_{l\to 1^+} \left[-\ln|1-x| \Big|_l^2 \right]$$

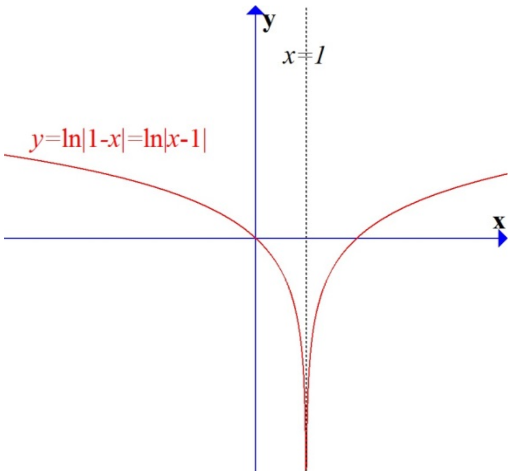
$$= \lim_{l\to 1^+} [-\ln|1-2| + \ln|1-l|]$$

$$= \lim_{l\to 1^+} [\ln|1-l|]$$

$$= -\infty$$

Therefore $\int_1^2 \frac{1}{1-x}\,dx$ diverges.





Example 3

Does the following improper integral converge or diverge? If it converges, evaluate the integral.

$$\int_{1}^{4} \frac{1}{(x-2)^{2/3}}\,dx$$

Example 3 (continued)

Solution:

$$\int_1^4 \frac{1}{(x-2)^{2/3}}\,dx = \int_1^2 \frac{1}{(x-2)^{2/3}}\,dx + \int_2^4 \frac{1}{(x-2)^{2/3}}\,dx$$

$$= \lim_{l\to 2^-} \int_1^l \frac{1}{(x-2)^{2/3}}\,dx + \lim_{l\to 2^+} \int_l^4 \frac{1}{(x-2)^{2/3}}\,dx$$

$$= \lim_{l\to 2^-} \left[3(x-2)^{1/3}\Big|_1^l\right] + \lim_{l\to 2^+} \left[3(x-2)^{1/3}\Big|_l^4\right]$$

$$= \lim_{l\to 2^-} \left[3(l-2)^{1/3} - 3(1-2)^{1/3}\right] + \lim_{l\to 2^+} \left[3(4-2)^{1/3} - 3(l-2)^{1/3}\right]$$

$$= [0+3] + \left[3\sqrt[3]{2} - 0\right] = 3 + 3\sqrt[3]{2}$$

Therefore $\int_1^4 \frac{1}{(x-2)^{2/3}}\,dx$ converges and $\int_1^4 \frac{1}{(x-2)^{2/3}}\,dx = 3 + 3\sqrt[3]{2}$.

Example 4

Does the following improper integral converge or diverge? If it converges, evaluate the integral.

$$\int_0^2 \frac{1}{(x-1)^2}\,dx$$

<u>Solution</u>:

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \frac{-1}{x-1}\Bigg|_0^2 = \frac{-1}{2-1} - \frac{-1}{0-1} = -2$$

Example 4 (continued)

BUT $\frac{1}{(x-1)^2} > 0$, so $\int_0^2 \frac{1}{(x-1)^2}\,dx > 0$. That means our answer $\int_0^2 \frac{1}{(x-1)^2}\,dx = -2$ is incorrect!

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{l\to 1^-} \int_0^l \frac{1}{(x-1)^2}\,dx + \lim_{l\to 1^+} \int_l^2 \frac{1}{(x-1)^2}\,dx$$

Example 4 (continued)

$$\lim_{l\to 1^-}\int_0^l \frac{1}{(x-1)^2}\,dx + \lim_{l\to 1^+}\int_l^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{l\to 1^-}\left(\frac{-1}{x-1}\Big|_0^l\right) + \lim_{l\to 1^+}\left(\frac{-1}{x-1}\Big|_l^2\right)$$

$$= \lim_{l\to 1^-}\left(\frac{-1}{l-1} - \frac{-1}{0-1}\right) + \lim_{l\to 1^+}\left(\frac{-1}{2-1} - \frac{-1}{l-1}\right)$$

Neither limit exists, so $\int_0^2 \frac{1}{(x-1)^2}\,dx$ diverges.

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