Improper Integrals

Part 1: Introduction and Integrals with Infinite Limits of Integration

Introduction

Right now

$$\int_0^5 \frac{1}{(x-2)^{2/3}}\,dx$$

does not exist because the integrand is not bounded on $[0,5]$.

$$\left(\lim_{x\to 2}(x-2)^{-2/3} = \infty\right)$$

Improper Integrals

We will extend definite integrals to include:

- Integrals with infinite limits of integration
- Integrals which are not bounded on the interval

These are called **improper integrals**.

Integrals with Infinite Limits of Integration

Let f be continuous on $[a, +\infty)$. Then

$$\int_a^{+\infty} f(x)\,dx = \lim_{l \to +\infty} \int_a^l f(x)\,dx$$

- If this limit exists, the improper integral is said to **converge**, and the value of the limit is the value of the integral.
- If this limit does not exist, then the improper integral **diverges**, and the integral has no value.

Integrals with Infinite Limits of Integration (continued)

Let f be continuous on $(-\infty, b]$. Then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{l \to -\infty} \int_{l}^{b} f(x)\,dx$$

- If this limit exists, the improper integral is said to **converge**, and the value of the limit is the value of the integral.
- If this limit does not exist, then the improper integral **diverges**, and the integral has no value.

Integrals with Infinite Limits of Integration (continued)

Let f be continuous on $(-\infty, +\infty)$. Then

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx$$

(Note: We often choose $c = 0$.)

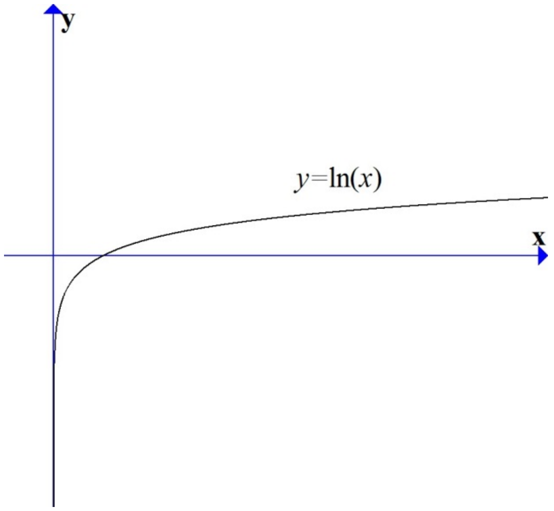
- If both integrals on the right exist, then $\int_{-\infty}^{+\infty} f(x)\,dx$ is said to **converge**.
- If either integral on the right does not exist, then $\int_{-\infty}^{+\infty} f(x)\,dx$ **diverges**.

Example 1

Does the following improper integral converge or diverge? If it converges, evaluate the integral.

$$\int_{1}^{+\infty} \frac{1}{x}\,dx$$

Example 1 (continued)



Solution:

$$\int_1^{+\infty} \frac{1}{x}dx = \lim_{l\to+\infty} \int_1^{l} \frac{1}{x}dx$$

$$= \lim_{l\to+\infty} \left[\ln|x| \Big|_1^l \right] = \lim_{l\to+\infty} [\ln|l| - \ln|1|]$$

$$= +\infty - 0 = +\infty$$

Therefore $\int_1^{+\infty} \frac{1}{x}dx$ diverges.

Example 2

Does the following improper integral converge or diverge? If it converges, evaluate the integral.

$$\int_{-\infty}^{0} e^{x}\, dx$$

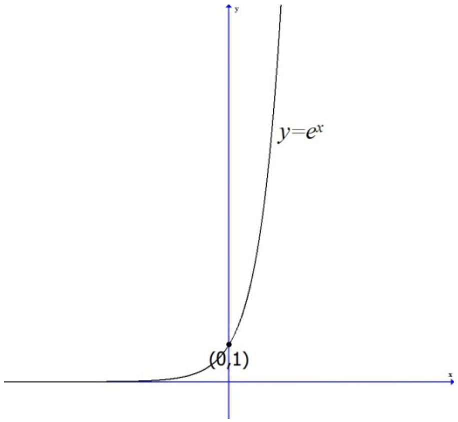
Example 2 (continued)

Solution:

$$\int_{-\infty}^{0} e^x\,dx = \lim_{l\to-\infty}\int_{l}^{0} e^x\,dx$$

$$= \lim_{l\to-\infty}\left[e^x\Big|_l^0\right] = \lim_{l\to-\infty}\left[e^0 - e^l\right]$$

$$= 1 - 0 = 1$$



Therefore $\int_{-\infty}^{0} e^x\,dx$ converges and $\int_{-\infty}^{0} e^x\,dx = 1$.

Example 3

Does the following improper integral converge or diverge? If it converges, evaluate the integral.

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$$

Example 3 (continued)

Solution:

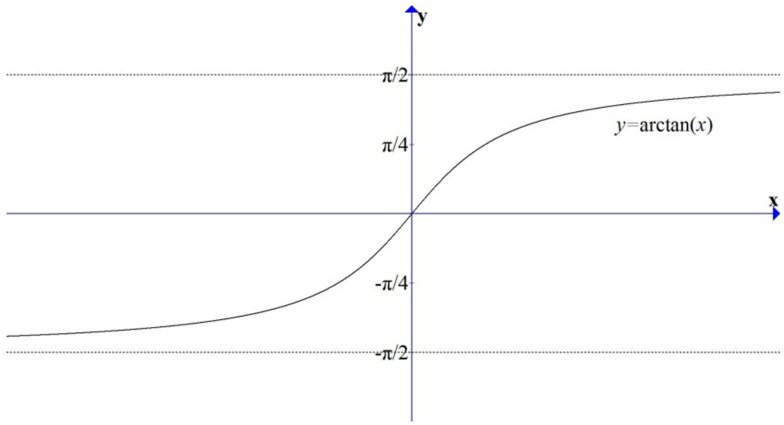
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{+\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{l\to-\infty} \int_{l}^{0} \frac{1}{1+x^2}\,dx + \lim_{l\to+\infty} \int_{0}^{l} \frac{1}{1+x^2}\,dx$$

$$= \lim_{l\to-\infty} \left[\tan^{-1} x \Big|_{l}^{0} \right] + \lim_{l\to+\infty} \left[\tan^{-1} x \Big|_{0}^{l} \right]$$

$$= \lim_{l\to-\infty} [\tan^{-1} 0 - \tan^{-1} l] + \lim_{l\to+\infty} [\tan^{-1} l - \tan^{-1} 0]$$

Example 3 (continued)

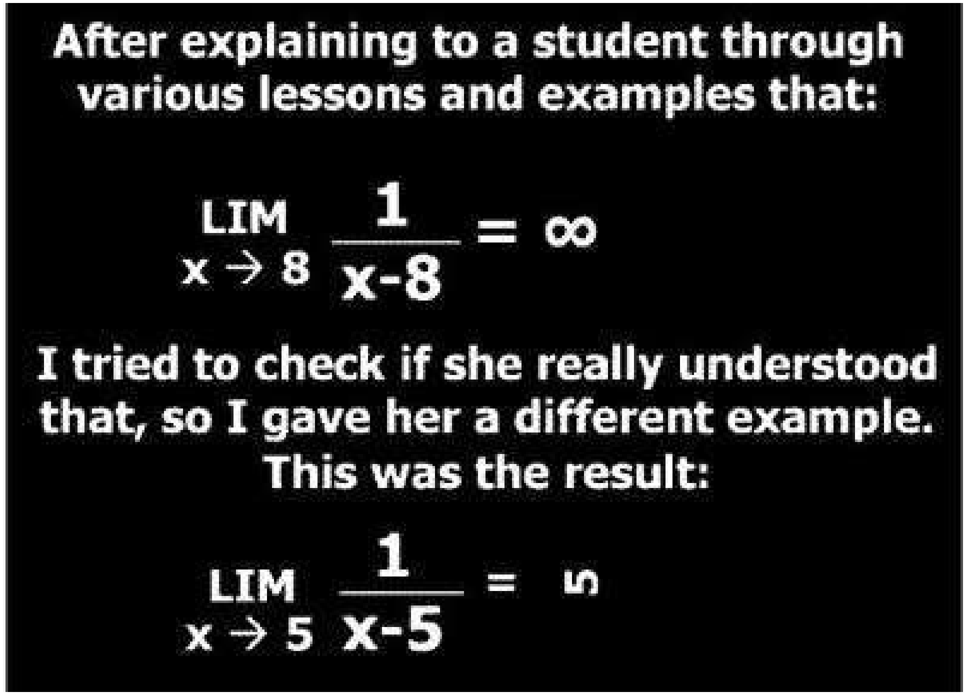


$$\lim_{l\to-\infty}[\tan^{-1}0-\tan^{-1}l]+\lim_{l\to+\infty}[\tan^{-1}l-\tan^{-1}0]$$

$$=\left[0-\left(-\frac{\pi}{2}\right)\right]-\left[\frac{\pi}{2}-0\right]=0$$

Therefore

$\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\,dx$ converges and $\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\,dx=0.$



http://www.calculushumor.com/3/category/limits/2.html