Integration of Rational Functions by Partial Fractions

Part 3:

Examples

Example 1

Evaluate

$$\int \frac{2x+4}{x^3-2x^2}\,dx$$

<u>Solution</u>:

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$$

Example 1 (continued)

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

Now, multiply both sides of the equation by $x^2(x-2)$ to get

$$2x + 4 = Ax(x-2) + B(x-2) + Cx^2$$

Multiply and collect like terms

$$2x + 4 = (A + C)x^2 + (-2A + B)x + (-2B)$$

Example 1 (continued)

$$2x + 4 = (A + C)x^2 + (-2A + B)x + (-2B)$$

From this we get the system of equations:

$$\begin{cases} A + C = 0 \\ -2A + B = 2 \\ -2B = 4 \end{cases}$$

Solving this system gives:

$$B = -2, A = -2, C = 2$$

Example 1 (continued)

Since

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$= \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$

we can now say

$$\int \frac{2x+4}{x^2(x-2)}dx = -2\int \frac{1}{x}dx - 2\int \frac{1}{x^2}dx + 2\int \frac{1}{x-2}dx$$

$$= -2\ln|x| - 2\left(-\frac{1}{x}\right) + 2\ln|x-2| + C$$

$$= \boxed{\frac{2}{x} + 2\ln\left|\frac{x-2}{x}\right| + C}$$

Example 2

Evaluate

$$\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\,dx$$

<u>Solution</u>:

$$\frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}$$

$$= \frac{x^2 + x - 2}{(3x^3 - x^2) + (3x - 1)}$$

$$= \frac{x^2 + x - 2}{x^2(3x - 1) + 1 \cdot (3x - 1)}$$

$$= \frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)}$$

Example 2 (continued)

$$\frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)} = \frac{A}{3x - 1} + \frac{Bx + C}{x^2 + 1}$$

Multiplying both sides by $(3x - 1)(x^2 + 1)$ gives:

$$\begin{aligned} x^2 + x - 2 &= A(x^2 + 1) + (Bx + C)(3x - 1) \\ &= Ax^2 + A + 3Bx^2 - Bx + 3Cx - C \\ &= (A + 3B)x^2 + (-B + 3C)x + (A - C) \end{aligned}$$

Example 2 (continued)

$$\begin{cases} A + 3B = 1 \\ -B + 3C = 1 \\ A - C = -2 \end{cases}$$

Subtracting the third row from the first gives

$$\begin{cases} A + 3B = 1 \\ -B + 3C = 1 \\ 3B + C = 3 \end{cases}$$

Adding 3 times the second row to the third gives

$$\begin{cases} A + 3B = 1 \\ -B + 3C = 1 \\ 10C = 6 \end{cases}$$

Example 2 (continued)

$$\begin{cases} A + 3B = 1 \\ -B + 3C = 1 \\ 10C = 6 \end{cases}$$

Solving the last row gives that $C = \frac{6}{10} = \frac{3}{5}$.

Substituting $C = \frac{3}{5}$ into the middle row gives

$$-B + 3\left(\frac{3}{5}\right) = 1 \Rightarrow B = \frac{4}{5}.$$

Substituting $B = \frac{4}{5}$ into the first row gives

$$A + 3\left(\frac{4}{5}\right) = 1 \Rightarrow A = -\frac{7}{5}.$$

Example 2 (continued)

Since

$$\frac{x^2+x-2}{(3x-1)(x^2+1)}=\frac{A}{3x-1}+\frac{Bx+C}{x^2+1}$$

$$=\frac{-\frac{7}{5}}{3x-1}+\frac{\frac{4}{5}x+\frac{3}{5}}{x^2+1}=\frac{-\frac{7}{5}}{3x-1}+\frac{\frac{4}{5}x}{x^2+1}+\frac{\frac{3}{5}}{x^2+1}$$

we have

$$\int\frac{x^2+x-2}{(3x-1)(x^2+1)}dx$$

$$=-\frac{7}{5}\int\frac{1}{3x-1}dx+\frac{4}{5}\int\frac{x}{x^2+1}dx+\frac{3}{5}\int\frac{1}{x^2+1}dx$$

Example 2 (continued)

$$\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\,dx$$

$$= -\frac{7}{5}\int \frac{1}{3x-1}\,dx + \frac{4}{5}\int \frac{x}{x^2+1}\,dx + \frac{3}{5}\int \frac{1}{x^2+1}\,dx$$

$$= -\frac{7}{5}\cdot\frac{1}{3}\ln|3x-1| + \frac{4}{5}\cdot\frac{1}{2}\ln|x^2+1| + \frac{3}{5}\tan^{-1}x + C$$

$$= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln(x^2+1) + \frac{3}{5}\tan^{-1}x + C$$

Example 3

Evaluate

$$\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2}\,dx$$

<u>Solution</u>:

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

Multiplying both sides by $(x+2)(x^2+3)^2$ gives:

Example 3 (continued)

Giving us:

$$3x^4 + 4x^3 + 16x^2 + 20x + 9$$
$$= A(x^2+3)^2 + (Bx+C)(x+2)(x^2+3)$$
$$+ (Dx+E)(x+2)$$
$$= (A+B)x^4 + (2B+C)x^3 + (6A+3B+2C+D)x^2$$
$$+ (6B+3C+2D+E)x + (9A+6C+2E)$$

We can set up a systems of equations to solve:

$$1A + 1B + 0C + 0D + 0E = 3$$
$$0A + 2B + 1C + 0D + 0E = 4$$
$$6A + 3B + 2C + 1D + 0E = 16$$
$$0A + 6B + 3C + 2D + 1E = 20$$
$$9A + 0B + 6C + 0D + 2E = 9$$

Possible to solve, but a bit tedious!

Example 3 (continued)

$$3x^4 + 4x^3 + 16x^2 + 20x + 9$$
$$= A(x^2+3)^2 + (Bx+C)(x+2)(x^2+3)$$
$$+ (Dx+E)(x+2)$$

To solve for the coefficients, substitute values of x to make the various terms zero.

Setting $x = -2$ gives us:

$$3(-2)^4 + 4(-2)^3 + 16(-2)^2 + 20(-2) + 9$$
$$= A\left((-2)^2+3\right)^2$$
$$+ (B(-2)+C)\left((-2)+2\right)\left((-2)^2+3\right)$$
$$+ (D(-2)+E)\left((-2)+2\right)$$

$$49 = 49A \Rightarrow A = 1$$

Example 3 (continued)

Now, looking back at the system of equations

$$1A + 1B = 3$$
$$2B + 1C = 4$$
$$6A + 3B + 2C + 1D = 16$$
$$6B + 3C + 2D + 1E = 20$$
$$9A + 6C + 2E = 9$$

since $A = 1$, you can see from the first row that $B = 2$.

The second row now gives $C = 0$.

The third row tells us that $D = 4$.

The fourth row gives us $E = 0$.

Example 3 (continued)

Since

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

$$= \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

we now have

Example 3 (continued)

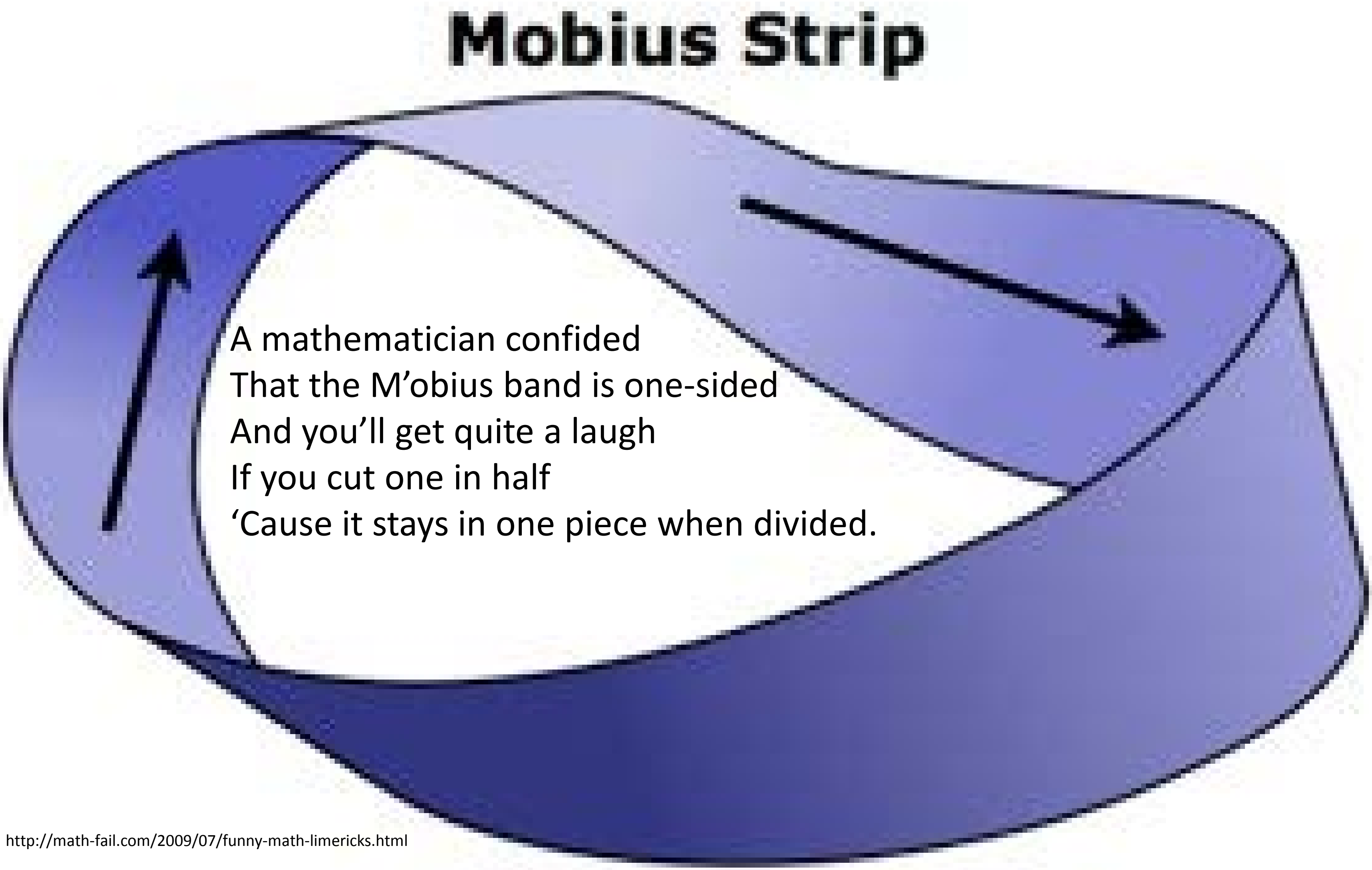
$$\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\,dx = \int \left(\frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2} \right) dx$$

$$= \int \frac{1}{x+2}\,dx + 2\int \frac{x}{x^2+3}\,dx + 4\int \frac{x}{(x^2+3)^2}\,dx$$

You can use a u-substitution of $u = x^2 + 3$ to solve the last two integrals.

$$= \ln|x+2| + 2 \cdot \frac{1}{2}\ln|x^2+3| + 4 \cdot (-1) \cdot \frac{1}{2} \cdot \frac{1}{x^2+3} + C$$

$$= \ln|x+2| + \ln(x^2+3) - \frac{2}{x^2+3} + C$$



A mathematician confided
That the M'obius band is one-sided
And you'll get quite a laugh
If you cut one in half
'Cause it stays in one piece when divided.

http://math-fail.com/2009/07/funny-math-limericks.html

http://www.thebookcast.com/indie-author-mathias-freese-this-mobius-strip-of-ifs/