Trigonometric Substitutions

Part 2: $\sqrt{x^2 + a^2}$

Example

Evaluate

$$\int \frac{dx}{\sqrt{x^2 + a^2}}$$

<u>Solution</u>:

This has an expression involving the form

$$\sqrt{x^2 + a^2}.$$

Example (continued)

We need x to take on every value between $-\infty$ and ∞.

Since $\tan\theta$ takes on every value exactly once on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, we will let

$$x = a\tan\theta\,, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Now, let's set up our substitutions and our reference triangle.

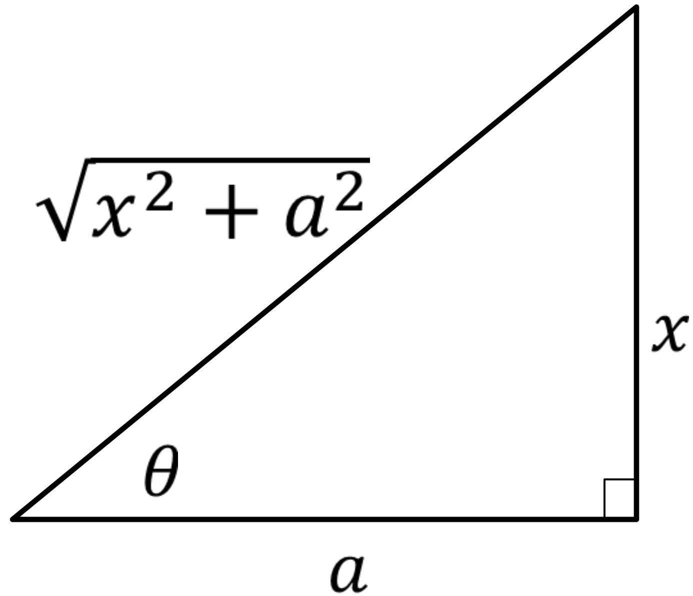
Example (continued)

$$x = a\tan\theta\,, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = a\sec^2\theta\, d\theta$$

$$\frac{x}{a} = \tan\theta = \frac{\text{OPP}}{\text{ADJ}}$$

$$\Rightarrow \tan^{-1}\left(\frac{x}{a}\right) = \theta$$



Example (continued)

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a \sec^2 \theta \, d\theta}{\sqrt{(a \tan \theta)^2 + a^2}}$$

$$= \int \frac{a \sec^2 \theta \, d\theta}{\sqrt{a^2 \tan^2 \theta + a^2}}$$

$$= \int \frac{a \sec^2 \theta \, d\theta}{\sqrt{a^2(\tan^2 \theta + 1)}}$$

$$= \int \frac{a \sec^2 \theta \, d\theta}{\sqrt{a^2 \sec^2 \theta}}$$

Example (continued)

$$\int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\sec^2\theta}} = \int \frac{a\sec^2\theta\, d\theta}{a\sec\theta}$$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

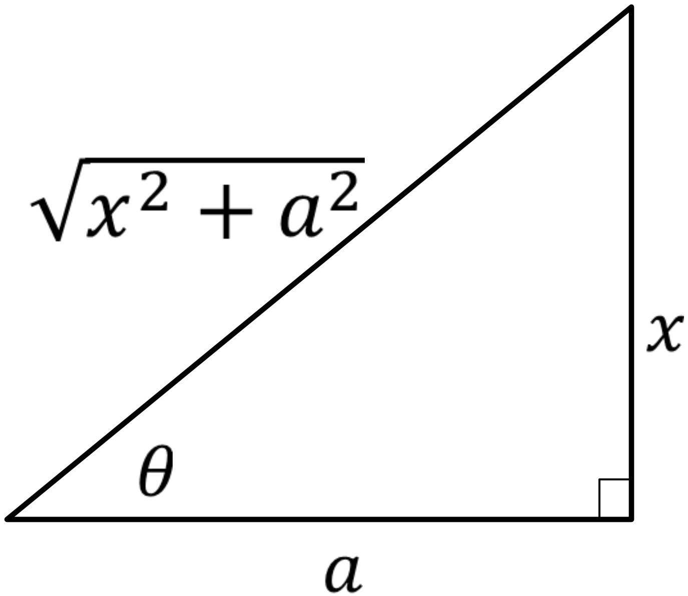
Now, going back to our reference triangle, we see

Example (continued)

$$x = a\tan\theta\,, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{x}{a}$$

$$\sec\theta = \frac{\text{HYP}}{\text{ADJ}} = \frac{\sqrt{x^2 + a^2}}{a}$$



Example (continued)

This gives

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right| + C$$

This is a fine answer, but we can make it better:

$$\ln\left|\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right| + C = \ln\left|\frac{\sqrt{x^2 + a^2} + x}{a}\right| + C$$

Example (continued)

$$\ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right|+C=\ln\left|\sqrt{x^2+a^2}+x\right|\underbrace{-\ln a}_{\text{constant}}+C$$

$$=\ln\left|\underbrace{\sqrt{x^2+a^2}+x}_{\text{positive for all } x}\right|+C$$

$$\int\frac{dx}{\sqrt{x^2+a^2}}=\boxed{\ln\left(\sqrt{x^2+a^2}+x\right)+C}$$

Lesson of the day

Don't be a

$$\frac{d^3 s}{dt^3}$$

http://math-fail.com/2013/11/lesson-of-the-day.html