Trigonometric Substitutions

Part 1:

Introduction and $\sqrt{a^2 - x^2}$

$$\sqrt{a^2 - x^2}, \sqrt{x^2 + a^2}, \sqrt{x^2 - a^2}$$

$(a > 0)$

In order to evaluate integrals involving the above expressions, we will eliminate the radicals by using an appropriate trigonometric substitution.

Example

Evaluate

$$\int \frac{dx}{x^2\sqrt{4 - x^2}}$$

<u>Solution</u>:

This has an expression involving the form

$$\sqrt{a^2 - x^2}$$

with $a = 2$.

Example (continued)

We need x to take on every value between $-a$ and a.

Since $\sin\theta$ takes on every value between -1 and 1 exactly once on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we will let

$$x = a\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Now, since $1 - \sin^2\theta = \cos^2\theta$, this gives us

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin\theta)^2}$$

Example (continued)

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin\theta)^2}$$
$$= \sqrt{a^2 - a^2\sin^2\theta}$$
$$= \sqrt{a^2(1 - \sin^2\theta)}$$
$$= \sqrt{a^2\cos^2\theta}$$
$$= |a\cos\theta|$$

And since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, we know that $\cos\theta \geq 0$, giving us

$$\sqrt{a^2 - x^2} = a\cos\theta$$

Example (continued)

$$\int \frac{dx}{x^2\sqrt{4-x^2}}$$

$$x = 2\sin\theta\,, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$
$$dx = 2\cos\theta\, d\theta$$

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-(2\sin\theta)^2}}$$

$$= \int \frac{2\cos\theta\, d\theta}{4\sin^2\theta \cdot 2\cos\theta}$$

Example (continued)

$$\int \frac{2\cos\theta\, d\theta}{4\sin^2\theta \cdot 2\cos\theta} = \int \frac{d\theta}{4\sin^2\theta}$$

$$= \int \frac{1}{4}\csc^2\theta\, d\theta$$

$$= \frac{1}{4}(-\cot\theta) + C = -\frac{1}{4}\cot\theta + C$$

We want the answer in terms of x!

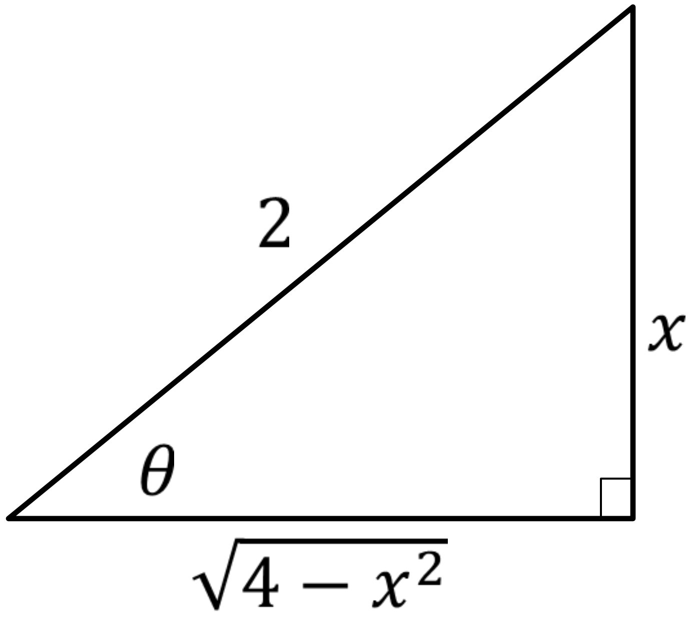
Example (continued)

$$x = 2\sin\theta\,, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\frac{x}{2} = \sin\theta = \frac{\text{OPP}}{\text{HYP}}$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{2}\right) = \theta$$

$$\cot\theta = \frac{\text{ADJ}}{\text{OPP}} = \frac{\sqrt{4-x^2}}{x}$$



Example (continued)

Taking the results from our reference triangle, we get

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\cot\theta + C$$

$$= -\frac{1}{4}\cdot\frac{\sqrt{4-x^2}}{x} + C$$

http://math-fail.com/2014/01/real-fact-812.html