Trigonometric Integrals

Part 2: Powers of Sine and Cosine

Integrating Power of Sine and Cosine

If m, n are non-negative integers, then

$$\int \sin^m x \cos^n x \, dx$$

can be evaluated by one of the following three procedures:

Integrating Power of Sine and Cosine

$$\int \sin^m x \cos^n x \, dx$$

Case	Procedure	Relevant Identities
n odd	Substitute $u = \sin x$	$\cos^2 x = 1 - \sin^2 x$
m odd	Substitute $u = \cos x$	$\sin^2 x = 1 - \cos^2 x$
n & m even	Use identities to reduce powers of sine and cosine	$\sin^2 x = \frac{1 - \cos 2x}{2}$ $\cos^2 x = \frac{1 + \cos 2x}{2}$

Example 1

Evaluate

$$\int \sin^4 x \cos^5 x \, dx$$

Solution:

Since the power of cosine is odd, we will "peel" off one power of cosine and re-write the even powers in terms of sine by using $\cos^2 x = 1 - \sin^2 x$.

Example 1 (continued)

$$\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cos^4 x \cos x \, dx$$

$$= \int \sin^4 x \, (\cos^2 x)^2 \cos x \, dx$$

$$= \int \sin^4 x \, (1 - \sin^2 x)^2 \cos x \, dx$$

$$u = \sin x$$
$$du = \cos x \, dx$$

Example 1 (continued)

$$\int \sin^4 x\,(1-\sin^2 x)^2 \cos x\,dx = \int u^4(1-u^2)^2\,du$$

$$= \int u^4(1-2u^2+u^4)\,du$$

$$= \int (u^4-2u^6+u^8)\,du$$

$$= \frac{1}{5}u^5 - 2\cdot\frac{1}{7}u^7 + \frac{1}{9}u^9 + C$$

$$= \boxed{\frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C}$$

Example 2

Evaluate

$$\int \sin^3 x \cos^2 x \, dx$$

Solution:

Since the power of sine is odd, we will "peel" off one power of sine and re-write the even powers in terms of cosine by using $\sin^2 x = 1 - \cos^2 x$.

Example 2 (continued)

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

$$u = \cos x$$
$$du = -\sin x \, dx \Rightarrow -du = \sin x \, dx$$

Example 2 (continued)

$$\int (1-\cos^2 x)\cos^2 x \sin x \, dx = \int (1-u^2)u^2\,(-\,du\,)$$

$$= -\int (u^2 - u^4)\,du$$

$$= -\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C$$

$$= \boxed{-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C}$$

Example 3

Evaluate

$$\int \sin^4 x \cos^4 x \, dx$$

<u>Solution</u>:

Since the power of both cosine and sine is even, we will re-write the even powers by using $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$.

Example 3 (continued)

$$\int \sin^4 x \cos^4 x \, dx = \int (\sin^2 x)^2 (\cos^2 x)^2 \, dx$$

$$= \int \left(\frac{1 - \cos 2x}{2}\right)^2 \left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \int \left[\left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)\right]^2 dx$$

$$= \int \left(\frac{1 - \cos^2 2x}{4}\right)^2 dx$$

$$= \int \left(\frac{\sin^2 2x}{4}\right)^2 dx$$

Example 3 (continued)

$$\int \left(\frac{\sin^2 2x}{4}\right)^2 dx = \int \frac{\sin^4 2x}{16} dx$$

$$u = 2x$$

$$du = 2dx \Rightarrow \frac{1}{2} du = dx$$

$$= \int \frac{\sin^4 u}{16} \left(\frac{1}{2} du\right)$$

$$= \frac{1}{32} \int \sin^4 u \, du$$

Example 3 (continued)

We saw in Example 2 of the previous video that

$$\int \sin^4 x\, dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Therefore,

$$\frac{1}{32}\int \sin^4 u\, du = \frac{1}{32}\left(\frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u\right) + C$$

$$= \frac{1}{32}\left(\frac{3}{8}\cdot 2x - \frac{1}{4}\sin(2\cdot 2x) + \frac{1}{32}\sin(4\cdot 2x)\right) + C$$

Example 3 (continued)

$$\frac{1}{32}\left(\frac{3}{8}\cdot 2x - \frac{1}{4}\sin(2\cdot 2x) + \frac{1}{32}\sin(4\cdot 2x)\right) + C$$

$$= \frac{1}{32}\left(\frac{3}{4}x - \frac{1}{4}\sin 4x + \frac{1}{32}\sin 8x\right) + C$$

$$= \boxed{\frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C}$$

$$\int \sin mx \cos nx \, dx \quad \int \sin mx \sin nx \, dx \quad \int \cos mx \cos nx \, dx$$

To compute integrals of the above forms, use:

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta))$$

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta))$$

Example 4

Evaluate

$$\int \sin 7x \cos 3x \, dx$$

<u>Solution</u>:

$$\sin 7x \cos 3x = \frac{1}{2}(\sin(7x - 3x) + \sin(7x + 3x))$$

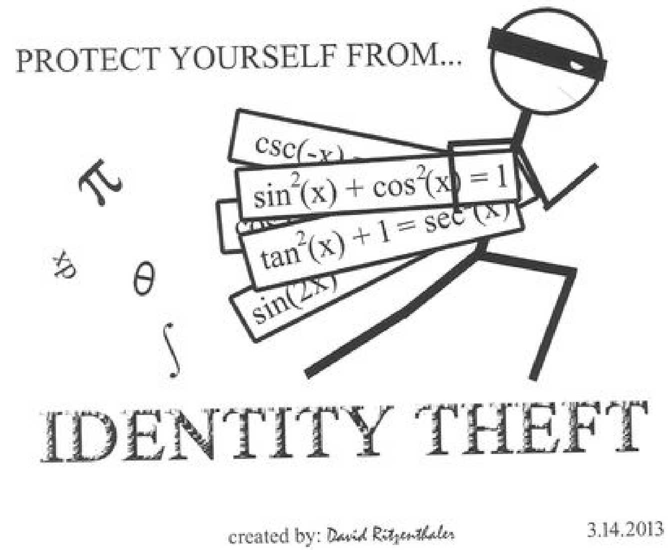
$$= \frac{1}{2}(\sin 4x + \sin 10x)$$

Example 4 (continued)

$$\int \sin 7x \cos 3x \, dx = \int \frac{1}{2}(\sin 4x + \sin 10x) \, dx$$

$$= \frac{1}{2}\left(-\frac{1}{4}\cos 4x - \frac{1}{10}\cos 10x\right) + C$$

$$= -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C$$



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