Arc Length

Part 1

Arc Length

Goal:

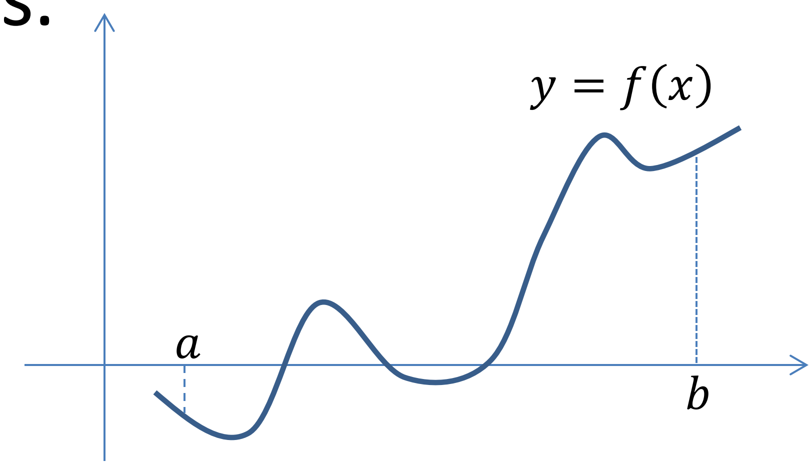
Find the arc length of a plane curve.

To start, we will consider only smooth curves.

Smooth

If a function f has a continuous derivative on an interval $[a, b]$ then f is **smooth**.

The graph of a smooth curve does not have any breaks, corners, or cusps.



Arc Length

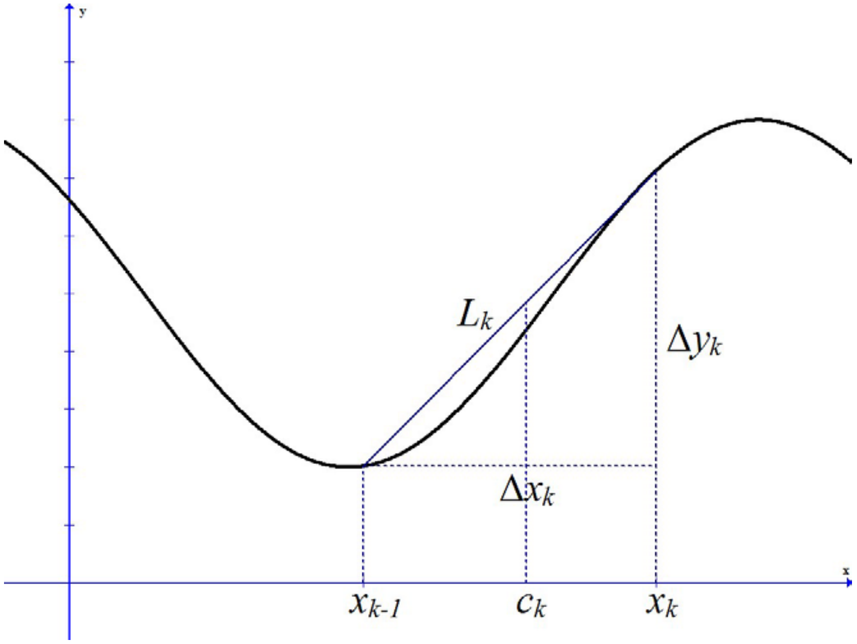
Let f be a smooth function on $[a, b]$,

$P = \{a = x_0, x_1, x_2, \cdots, x_{n-1}, x_n = b\}$ be a partition of $[a, b]$

and let L be how long the curve is from $x = a$ to $x = b$.

Then, on the k-th subinterval of $[a, b]$ we have:

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$



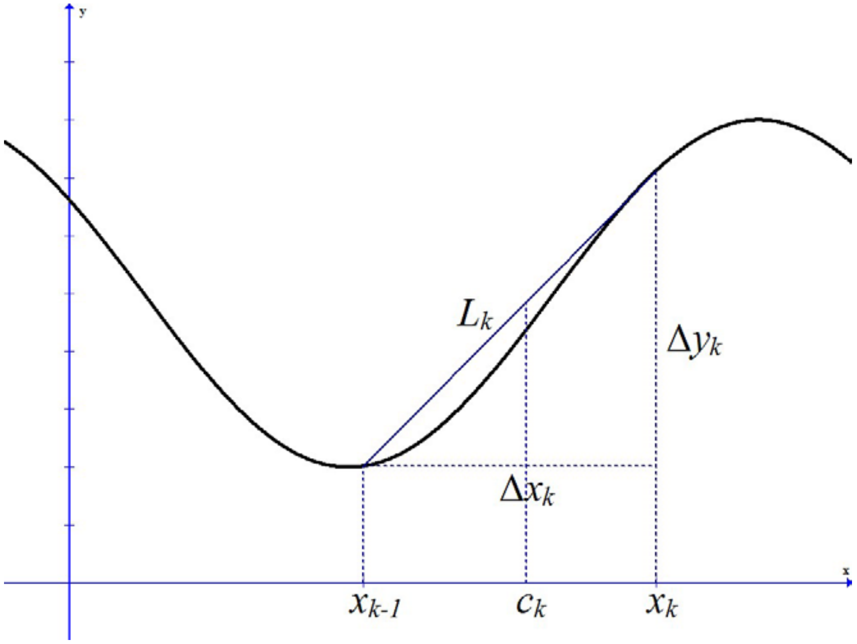
Arc Length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$= \sqrt{(\Delta x_k)^2 \left(1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right)}$$

$$= \Delta x_k \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}$$

$$\approx \Delta x_k \sqrt{1 + (f'(c_k))^2}$$



Arc Length

$$L \approx \sum_{k=1}^{n} \Delta x_k \sqrt{1 + (f'(c_k))^2}$$

$$= \sum_{k=1}^{n} \sqrt{1 + (f'(c_k))^2} \cdot \Delta x_k$$

This is just a Riemann sum for the function $\sqrt{1 + (f'(x))^2}$ on the interval $[a, b]$.

Definition of Arc Length

Let f be a smooth function on $[a, b]$. Then the **arc length** of $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Similarly, if $x = g(y)$ is a smooth function on $[c, d]$, then the **arc length** of $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

Example

Find the arc length of the curve $y = x^{3/2}$ from the point $(1,1)$ to the point $\left(2,2\sqrt{2}\right)$.

<u>Solution</u>:

We will solve this in two ways:

Example – Method 1

Method 1: from $(1,1)$ to $\left(2,2\sqrt{2}\right)$

$$y = x^{3/2} \Longrightarrow \frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

$$= \int_1^2 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\,dx$$

$$= \int_1^2 \sqrt{1 + \frac{9}{4}x}\,dx$$

Example – Method 1 (continued)

$$L = \int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$u = 1 + \frac{9}{4}x$$

$$du = \frac{9}{4}dx \Rightarrow \frac{4}{9}du = dx$$

$$x = 2 \Rightarrow u = 1 + \frac{9}{4} \cdot 2 = \frac{11}{2}$$

$$x = 1 \Rightarrow u = 1 + \frac{9}{4} \cdot 1 = \frac{13}{4}$$

Example – Method 1 (continued)

$$L = \int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$= \int_{13/4}^{11/2} \sqrt{u} \cdot \frac{4}{9} du$$

$$= \cdots = \frac{22\sqrt{22} - 13\sqrt{13}}{27}$$

Example – Method 2

Method 2: from $(1,1)$ to $\left(2,2\sqrt{2}\right)$

$$y = x^{3/2} \Longrightarrow x = y^{2/3} \Longrightarrow \frac{dx}{dy} = \frac{2}{3}y^{-1/3}$$

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

$$= \int_1^{2\sqrt{2}} \sqrt{1 + \left(\frac{2}{3}y^{-1/3}\right)^2}\, dy$$

$$= \int_1^{2\sqrt{2}} \sqrt{1 + \frac{4}{9}y^{-2/3}}\, dy$$

Example – Method 2 (continued)

$$L = \int_1^{2\sqrt{2}} \sqrt{y^{-2/3}\left(y^{2/3} + \frac{4}{9}\right)}\, dy$$

$$= \int_1^{2\sqrt{2}} y^{-1/3} \sqrt{y^{2/3} + \frac{4}{9}}\, dy$$

$$u = y^{2/3} + \frac{4}{9}$$

$$du = \frac{2}{3} y^{-1/3} dy \Rightarrow \frac{3}{2} du = y^{-1/3} dy$$

$$y = 2\sqrt{2} \Rightarrow u = \left(2\sqrt{2}\right)^{2/3} + \frac{4}{9} = \left(2^{3/2}\right)^{2/3} + \frac{4}{9} = 2 + \frac{4}{9} = \frac{22}{9}$$

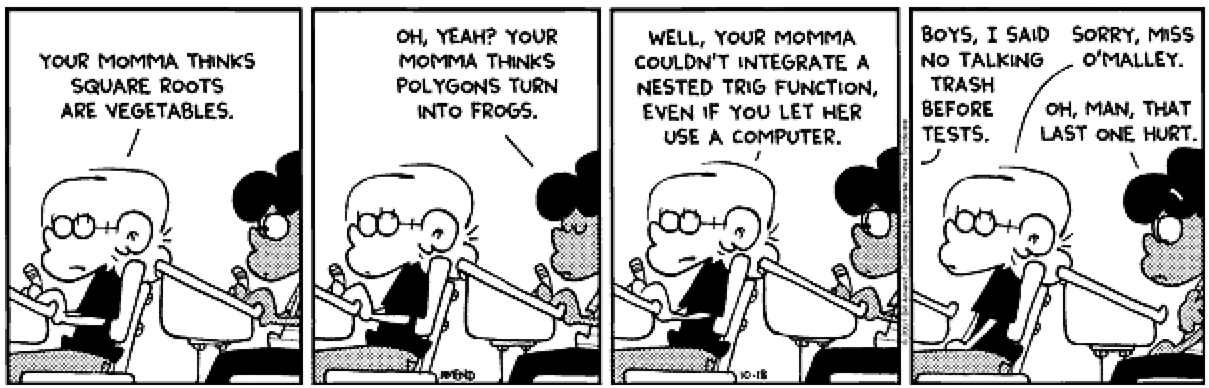
$$y = 1 \Rightarrow u = (1)^{2/3} + \frac{4}{9} = 1 + \frac{4}{9} = \frac{13}{9}$$

Example – Method 2 (continued)

$$L = \int_{1}^{2\sqrt{2}} y^{-1/3} \sqrt{y^{2/3} + \frac{4}{9}}\, dy$$

$$= \int_{13/9}^{22/9} \sqrt{u} \cdot \frac{3}{2}\, du$$

$$= \cdots = \frac{22\sqrt{22} - 13\sqrt{13}}{27}$$



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