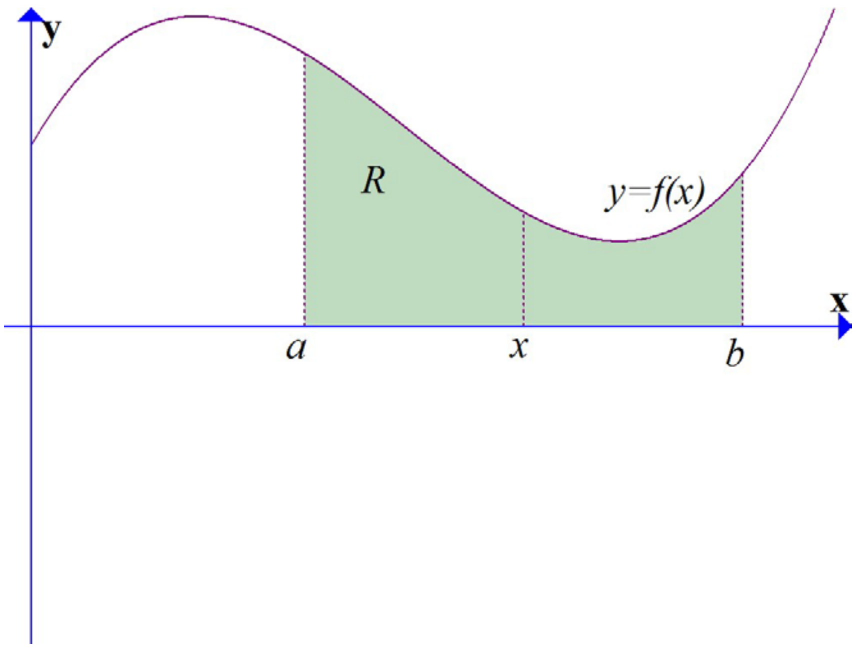
Volumes Using Cross-Sections

Part 2
Disk Method
Washer Method

Volumes of Solids of Revolutions

Let f be non-negative and continuous on $[a, b]$ and let R be the region bounded above by the graph $y = f(x)$ and below by the x-axis and on the sides by the lines $x = a$ and $x = b$.



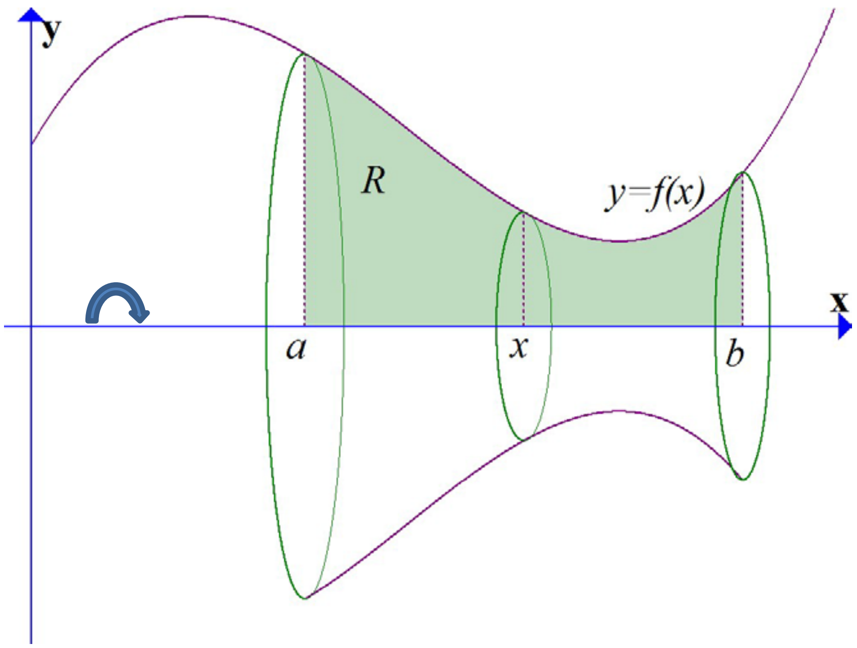
Volumes of Solids of Revolutions

Revolve this region about the x-axis to generate a solid having circular cross sections.

Since the cross section at x has radius $f(x)$,

the cross-sectional area is

$$A(x) = \pi[\text{radius}]^2$$
$$A(x) = \pi[f(x)]^2$$



Method of Disks

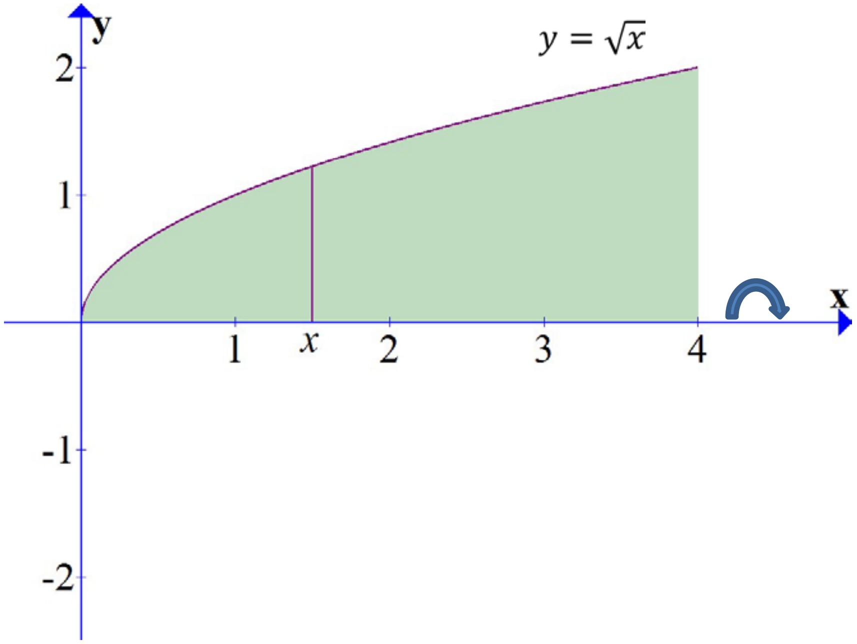
Let f be non-negative and continuous on $[a, b]$ and let R be the region bounded above by the graph $y = f(x)$ and below by the x-axis and on the sides by the lines $x = a$ and $x = b$.

Then the volume of the solid generated by revolving this region about the x-axis is

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[f(x)]^2\,dx$$

Example 1

Find the volume of the solid obtained when the region under $y = \sqrt{x}$ over $[0,4]$ is revolved about the x-axis.

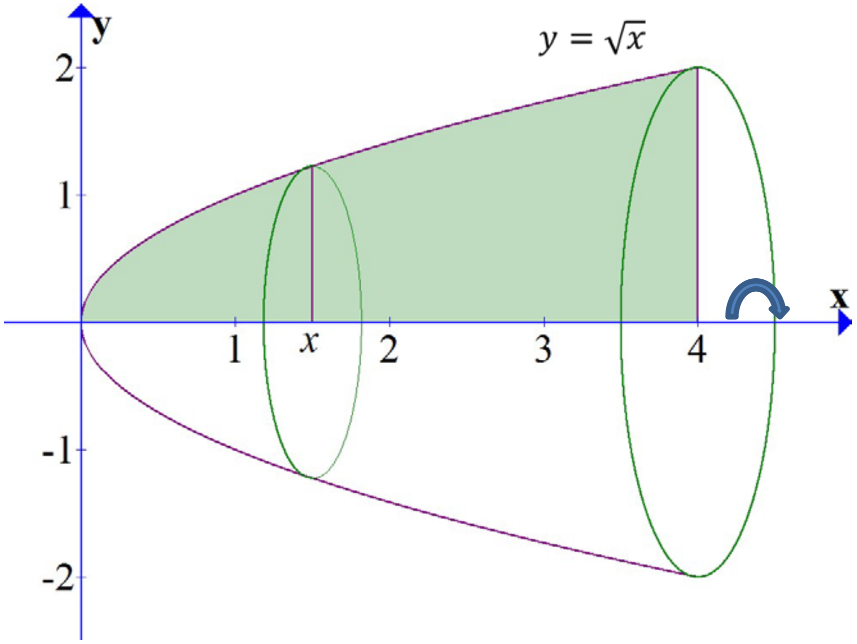


Solution:

First sketch the region to be revolved, draw the radius (slice) at x and indicate the direction of rotation.

Example 1 (continued)

$$V = \int_a^b \pi[f(x)]^2\,dx$$
$$= \int_0^4 \pi[\sqrt{x}]^2\,dx$$
$$= \int_0^4 \pi x\,dx$$
$$= \frac{\pi}{2}x^2\Big|_0^4$$
$$= \frac{\pi}{2}\cdot 4^2 - \frac{\pi}{2}\cdot 0^2 = 8\pi$$

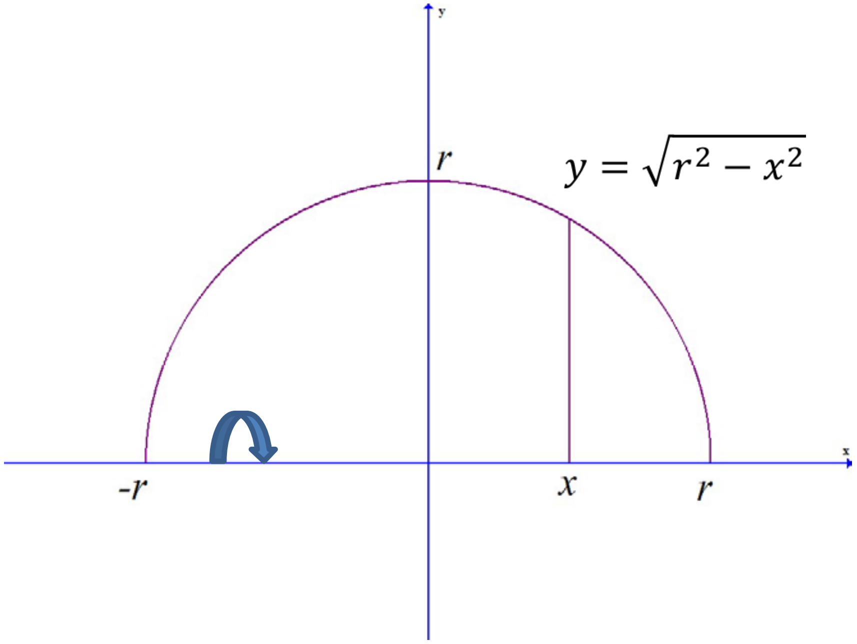


Example 2

Derive the formula for the volume of a sphere of radius r.

Solution:

A sphere is the upper semi-circle revolved about the x-axis.



Example 2 (continued)

$$V = \int_a^b \pi[f(x)]^2\,dx$$

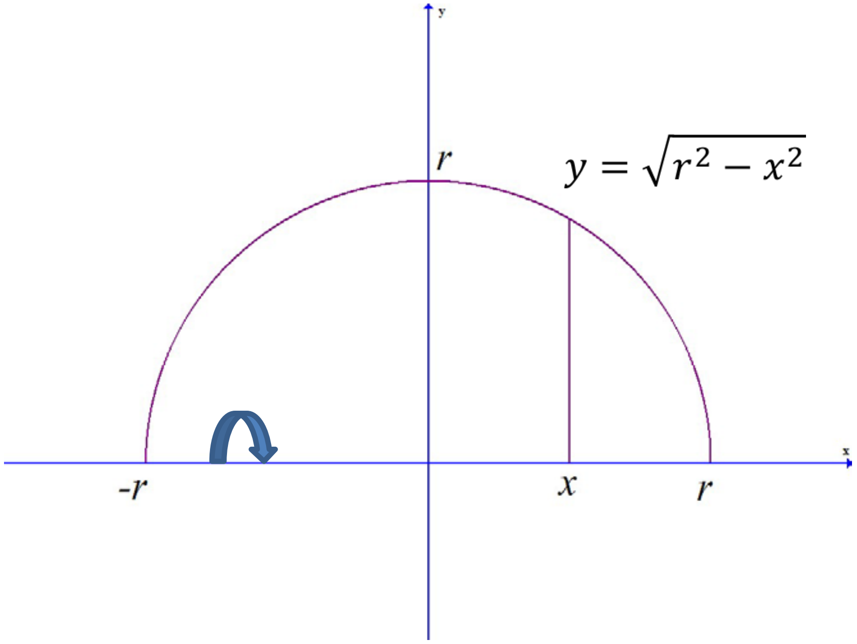
$$= \int_{-r}^{r} \pi\left[\sqrt{r^2 - x^2}\right]^2 dx$$

$$= 2\int_0^r \pi(r^2 - x^2)\,dx$$

$$= 2\pi\left(r^2x - \frac{1}{3}x^3\right)\Big|_0^r$$

$$= 2\pi\left(r^2r - \frac{1}{3}r^3\right) - 2\pi\left(r^2\cdot 0 - \frac{1}{3}\cdot 0^3\right)$$

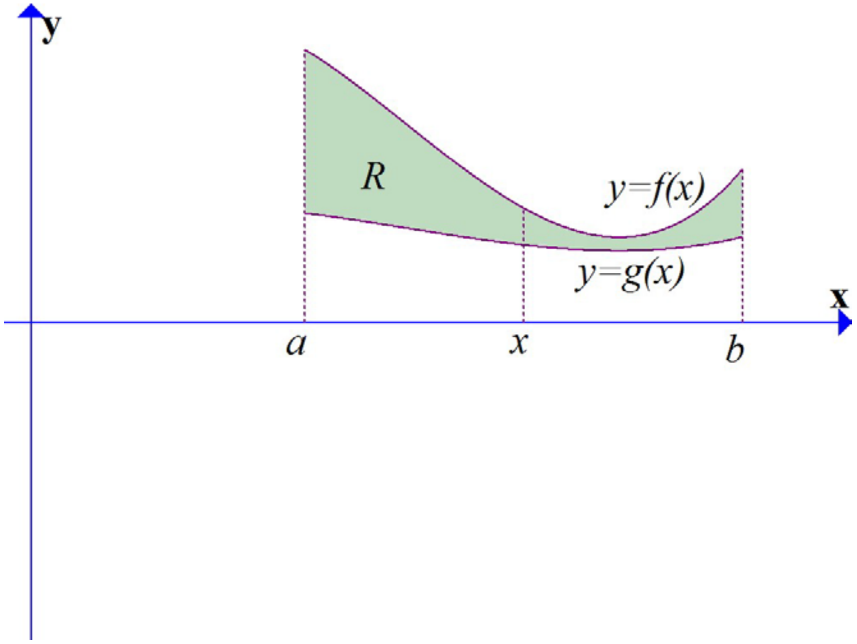
$$= \frac{4}{3}\pi r^3$$



Volumes of Solids of Revolutions

Suppose f and g are nonnegative continuous functions such that $g(x) \leq f(x)$ on $[a, b]$.

Let R be the region enclosed between the graphs of these functions and the lines $x = a$ and $x = b$.



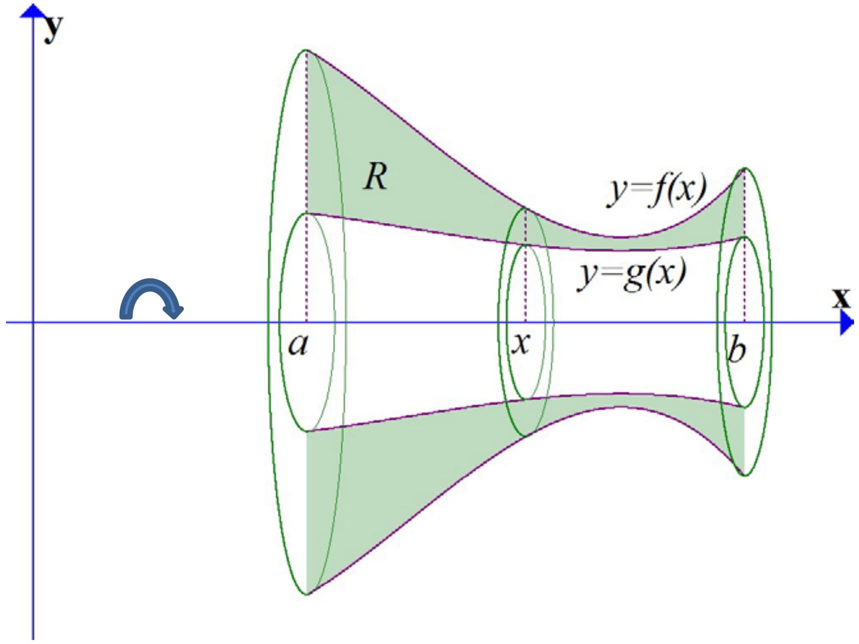
Volumes of Solids of Revolutions

Revolve this region about the x-axis.

Since the cross section at x has inner radius $g(x)$ and outer radius $f(x)$,

its area is

$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi([f(x)]^2 - [g(x)]^2)$$



Method of Washers

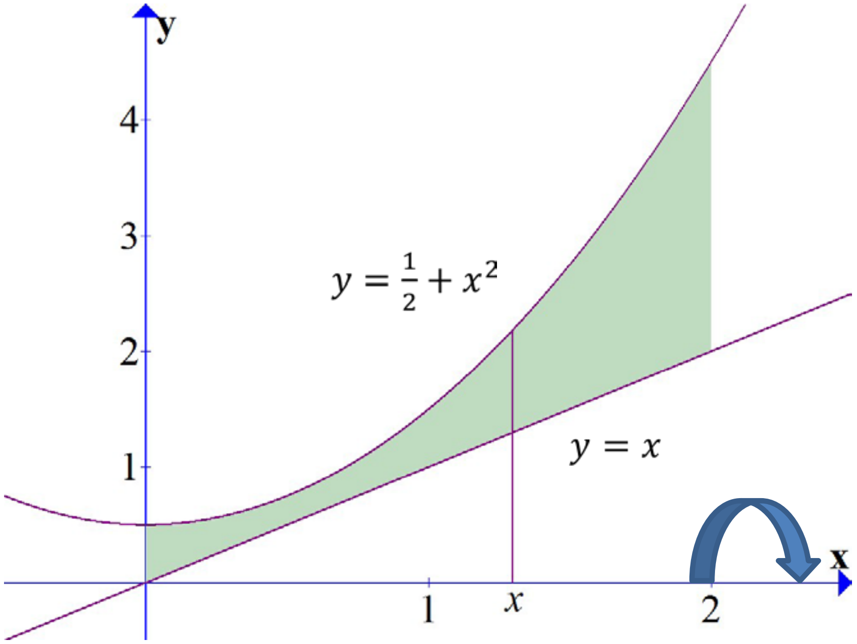
Let f and g be non-negative and continuous with $g(x) \leq f(x)$ on $[a, b]$ and let R be the region bounded above by the graph $y = f(x)$ and below by the graph $y = g(x)$ and on the sides by the lines $x = a$ and $x = b$.

Then the volume of the solid generated by revolving this region about the x-axis is

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right) dx$$

Example 3

Find the volume of the solid generated when the region between the graphs of $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over $[0,2]$ is revolved about the x-axis.



Solution:

First, sketch the region.

Example 3 (continued)

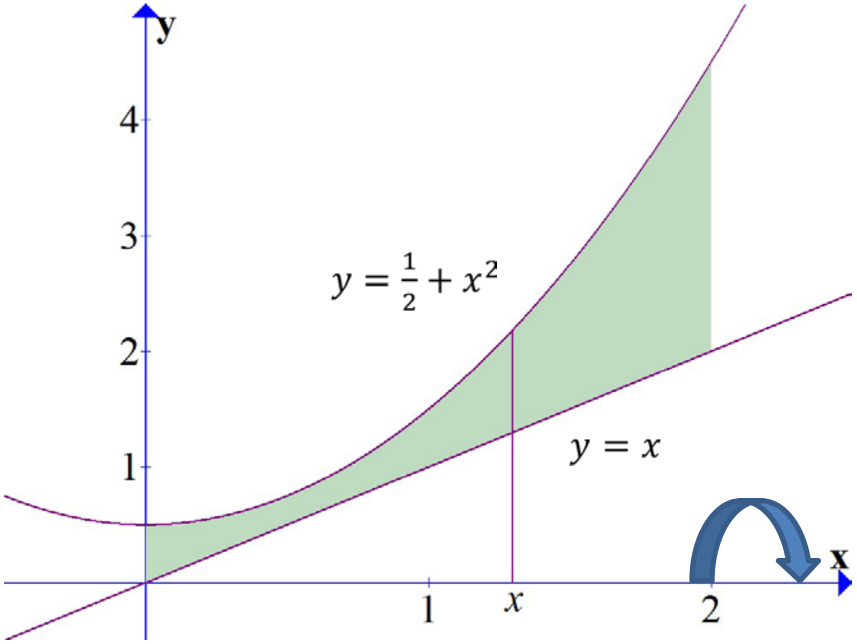
$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx$$

$$= \int_0^2 \pi\left(\left[\frac{1}{2} + x^2\right]^2 - [x]^2\right) dx$$

$$= \int_0^2 \pi\left(\frac{1}{4} + x^2 + x^4 - x^2\right) dx$$

$$= \int_0^2 \pi\left(\frac{1}{4} + x^4\right) dx$$

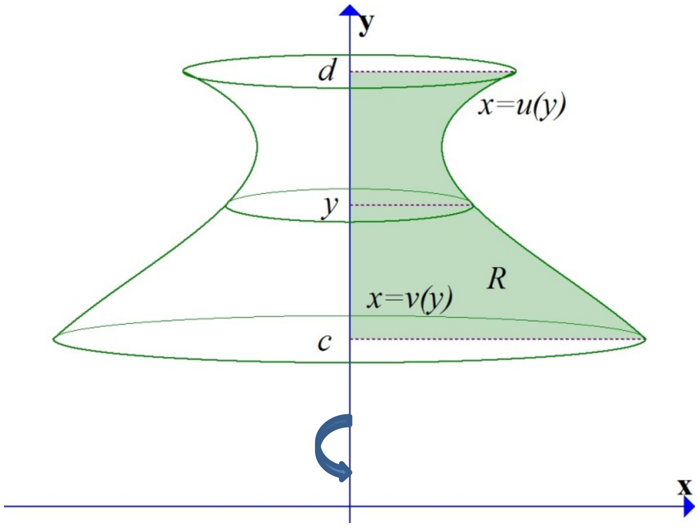
$$= \cdots = \frac{69\pi}{10}$$



Revolving about the y-axis

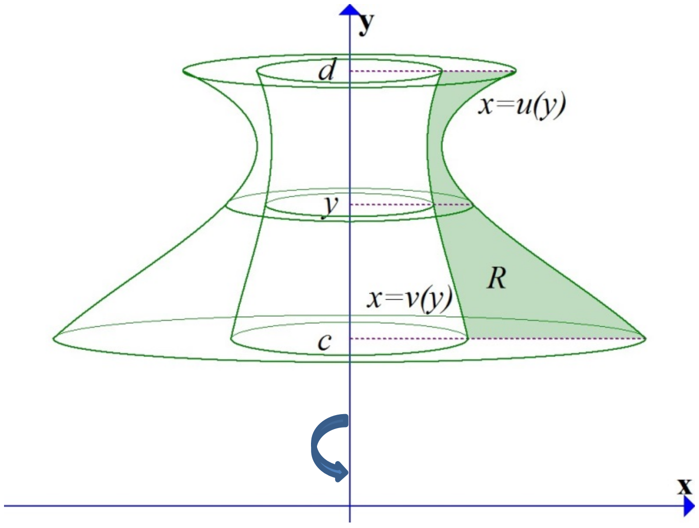
Disk Method:

$$V = \int_c^d \pi[u(y)]^2\, dy$$



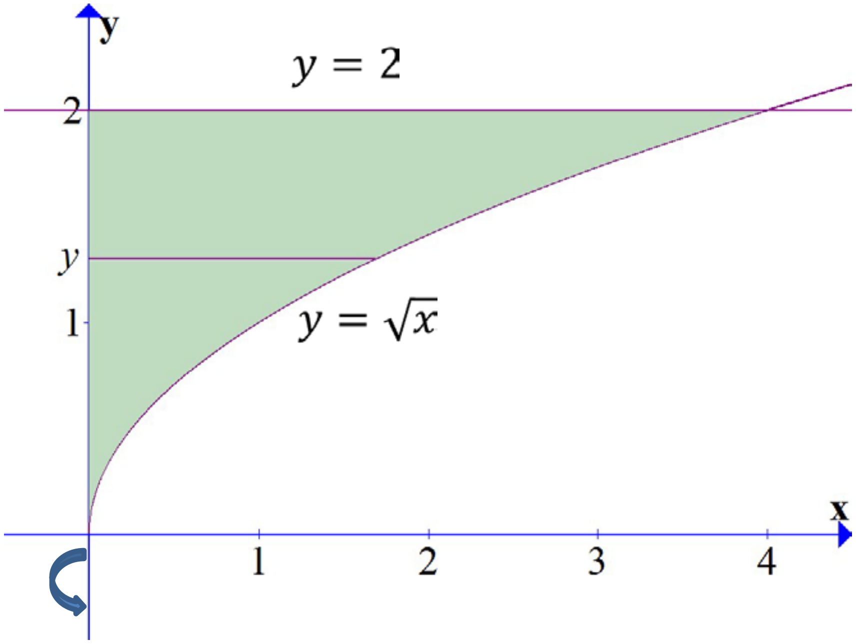
Washer Method:

$$V = \int_c^d \pi([u(y)]^2 - [v(y)]^2)\, dy$$



Example 4

Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis.



Solution:

First, sketch the region to be revolved.

Example 4 (continued)

Since we are revolving about the y-axis, we need to re-write the equations as functions of y.

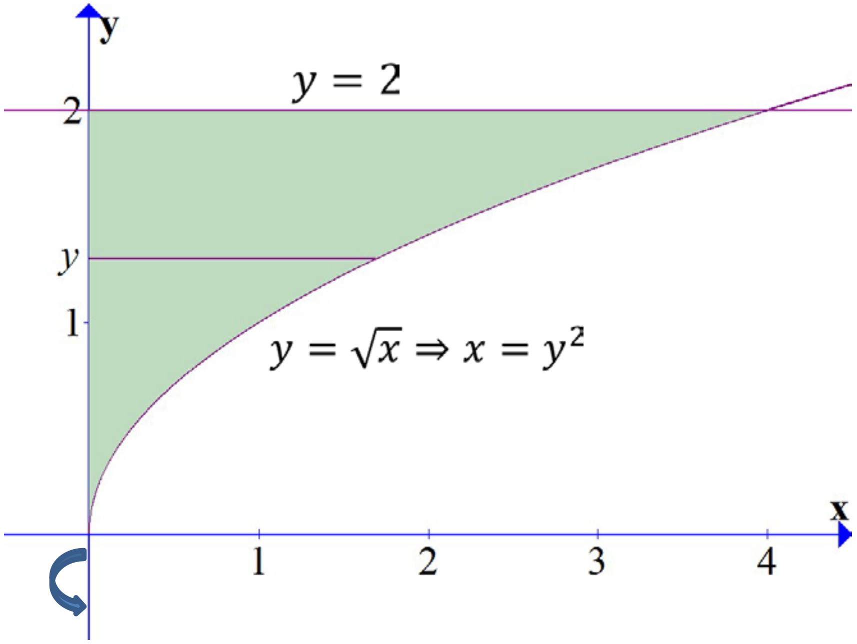
$$y = \sqrt{x} \Rightarrow x = y^2$$

Notice that this is a disk method problem.

$$V = \int_c^d \pi[u(y)]^2\, dy$$

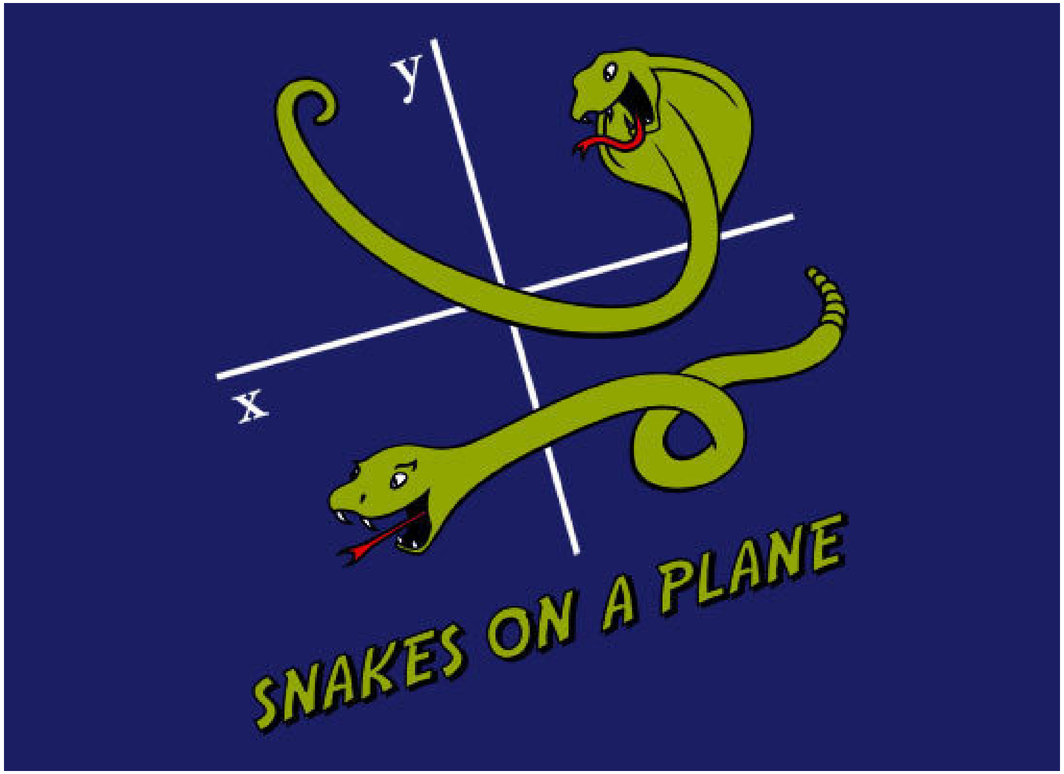
$$= \int_0^2 \pi[y^2]^2\, dy = \int_0^2 \pi y^4\, dy$$

$$= \cdots = \frac{32\pi}{5}$$



Disk/Washer Method Hints

- Always sketch the graphs, noting any x-intercepts and intersections
- If revolving about the x-axis:
  - The slices are perpendicular to the x-axis
  - Integrate with respect to x
- If revolving about the y-axis:
  - The slices are perpendicular to the y-axis
  - Integrate with respect to y



http://shirtshovel.com/math-snakesonaplane.shtml