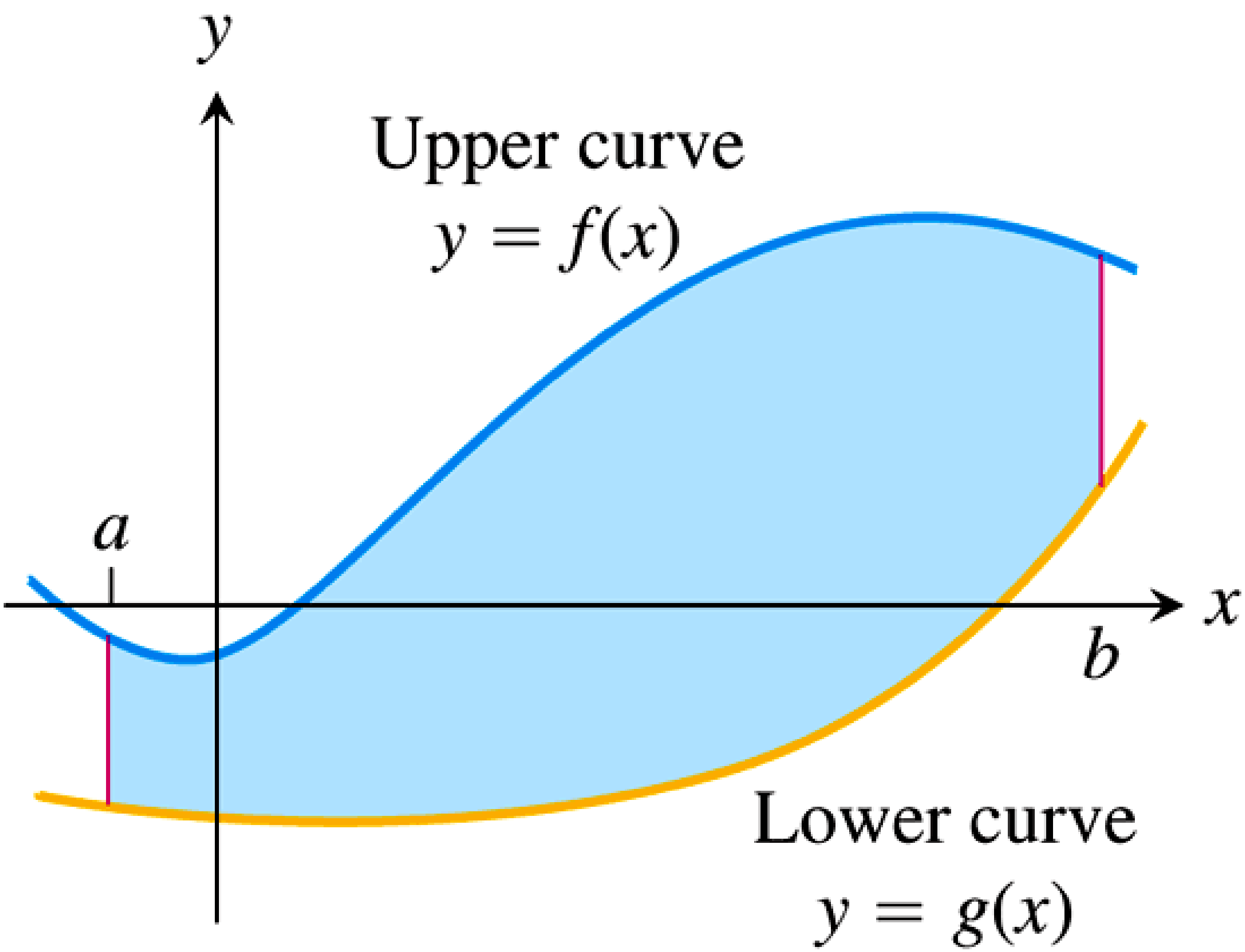
Substitution and Area Between Curves

Part 2: Area Between Two Curves



y
Upper curve
y = f(x)
a
x
b
Lower curve
y = g(x)

Area Between Two Curves

If f and g are continuous with $f(x) \geq g(x)$ on $[a, b]$,

then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$ and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

We call A the **area of the region between $\boldsymbol{y = f(x)}$ and$\boldsymbol{y = g(x)}$ from $\boldsymbol{a}$ to $\boldsymbol{b}$.**

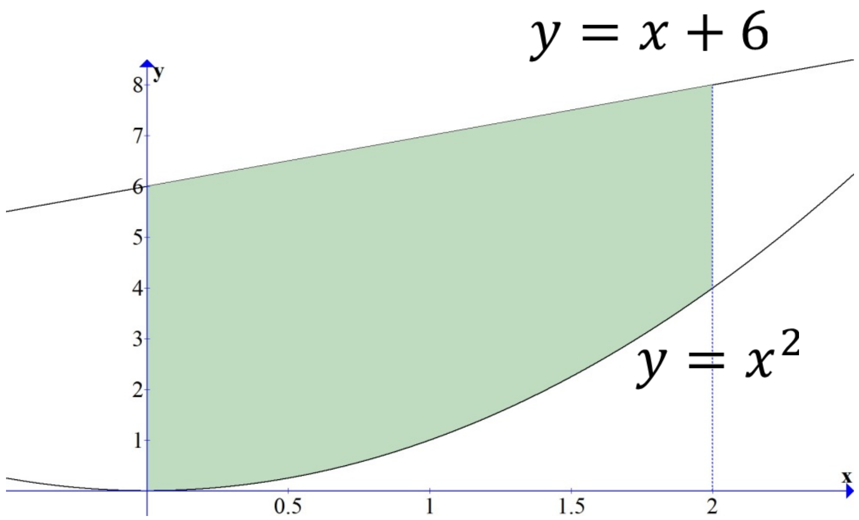
Example 1

Find the area of the region between $y = x + 6$ and $y = x^2$ from 0 to 2.

<u>Solution</u>:

To determine the top function, the bottom function, and the limits of integration, it is often helpful to make a sketch.

Example 1 (continued)



$$A = \int_0^2 [(x+6) - x^2]\,dx$$

$$= \left(\frac{1}{2}x^2 + 6x - \frac{1}{3}x^3\right)\Big|_0^2$$

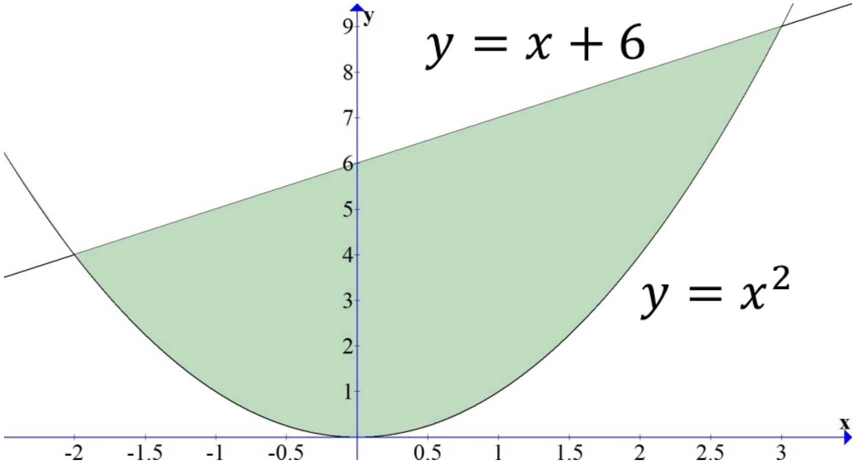
$$= \left(\frac{1}{2}\cdot 2^2 + 6\cdot 2 - \frac{1}{3}\cdot 2^3\right)$$

$$- \left(\frac{1}{2}\cdot 0^2 + 6\cdot 0 - \frac{1}{3}\cdot 0^3\right)$$

$$= \frac{34}{3}$$

Example 2

Find the area of the region enclosed between $y = x + 6$ and $y = x^2$.



<u>Solution</u>:

First, find where the two curves meet:

$$x^2 = x + 6$$
$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x = 3, x = -2$$

Example 2 (continued)

$$A = \int_{-2}^{3} [(x+6) - x^2]\, dx$$

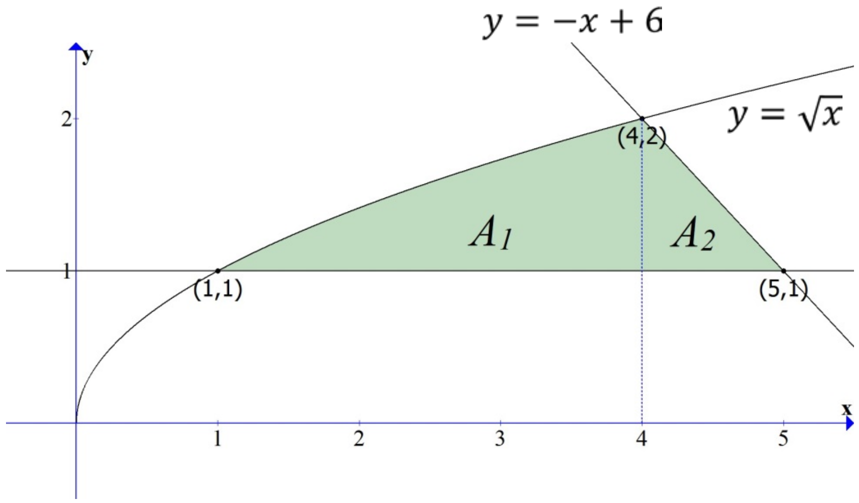
$$= \left(\frac{1}{2} x^2 + 6x - \frac{1}{3} x^3 \right) \Bigg|_{-2}^{3}$$

$$= \left(\frac{1}{2} \cdot 3^2 + 6 \cdot 3 - \frac{1}{3} \cdot 3^3 \right)$$

$$- \left(\frac{1}{2} \cdot (-2)^2 + 6 \cdot (-2) - \frac{1}{3} \cdot (-2)^3 \right) = \frac{125}{6}$$

Example 3

Find the area of the region enclosed by the curves $y = \sqrt{x}$, $y = -x + 6$ and $y = 1$.



<u>Solution</u>:

First sketch the curves, clearly labeling the intersection points.

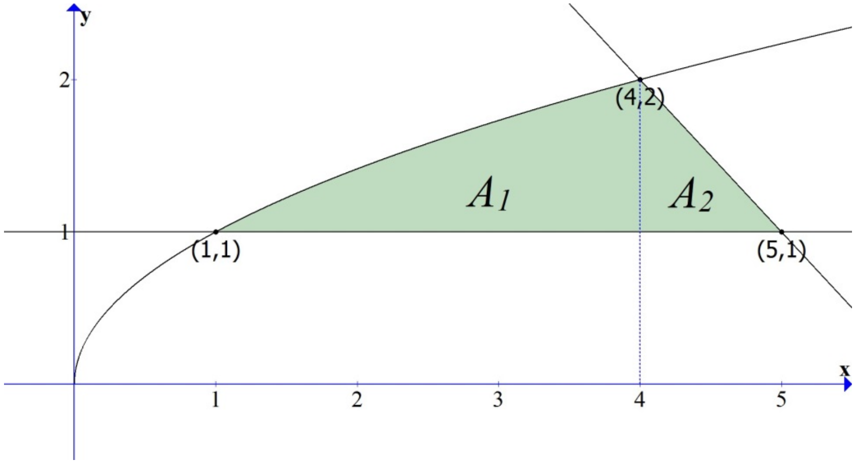
Example 3 (continued)

Notice that we need to break this up into two integrals:

$$A_1 = \int_1^4 [\sqrt{x} - 1]\, dx = \cdots = \frac{5}{3}$$

$$A_2 = \int_4^5 [(-x+6) - 1]\, dx$$

$$= \cdots = \frac{1}{2}$$

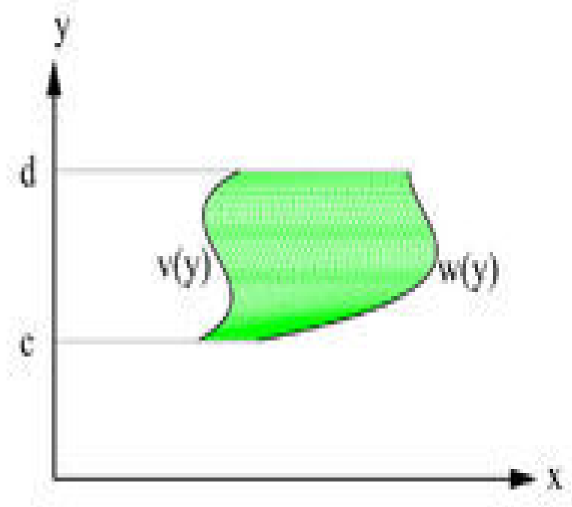


<u>Answer</u>:

$$A = A_1 + A_2 = \frac{5}{3} + \frac{1}{2} = \frac{13}{6}$$

Area by Integrating with Respect to y

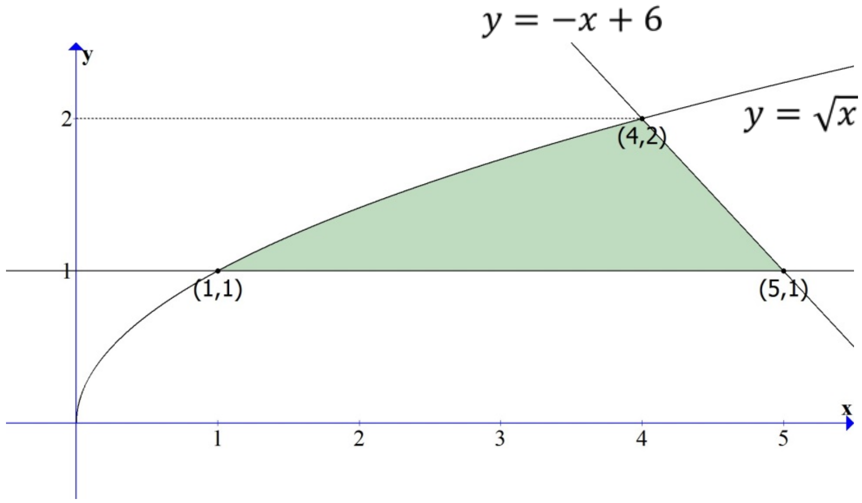
Sometimes you need to find the area of a region bounded above and below by horizontal lines and bounded on the left and right by the graphs of two functions of y.



Example 4

Repeat Example 3 by integrating with respect to y.

(Find the area of the region enclosed by the curves $y = \sqrt{x}$, $y = -x + 6$ and $y = 1$.)



Example 4 (continued)

Solution:

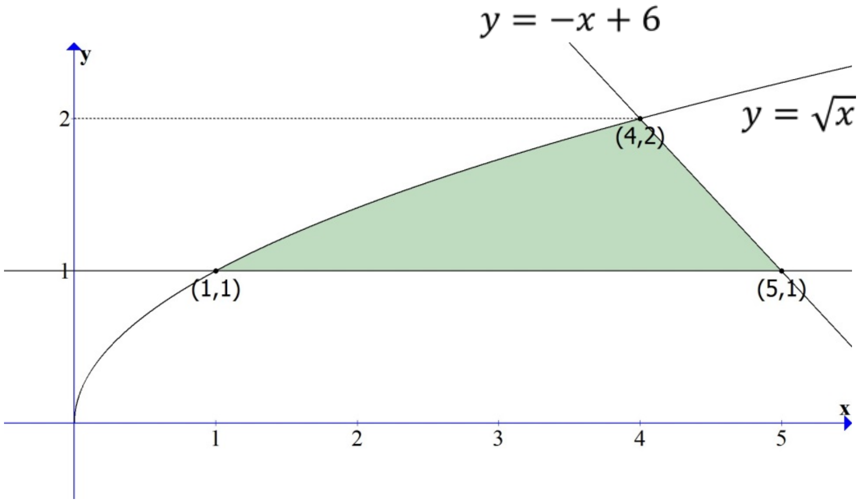
$$y = -x + 6 \Rightarrow x = -y + 6$$

$$y = \sqrt{x} \Rightarrow x = y^2$$

$$A = \int_1^2 [(-y + 6) - y^2]\, dy$$

$$= \cdots = \frac{13}{6}$$

This was much easier since we did not need to calculate two different integrals to find the area.



Example 5

Find the area of the region enclosed by the curves $y = x^4 - 4x^2 + 4$ and $y = x^2$.

<u>Solution</u>:

Clearly, $y = x^2$ is a parabola with vertex (0,0) that opens upwards.

$$\begin{aligned} y &= x^4 - 4x^2 + 4 \\ &= (x^2 - 2)^2 \\ &= \left(x - \sqrt{2}\right)^2 \left(x + \sqrt{2}\right)^2 \end{aligned}$$

Example 5 (continued)

So the only x-intercepts are $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, 0\right)$.

Since $y = x^4 - 4x^2 + 4$ is a 4th degree polynomial with a positive leading coefficient,

we know that the shape of the curve will look like a rounded "W".

Since the graph only crosses the x-axis at two points, they must be the bottoms of the "W".

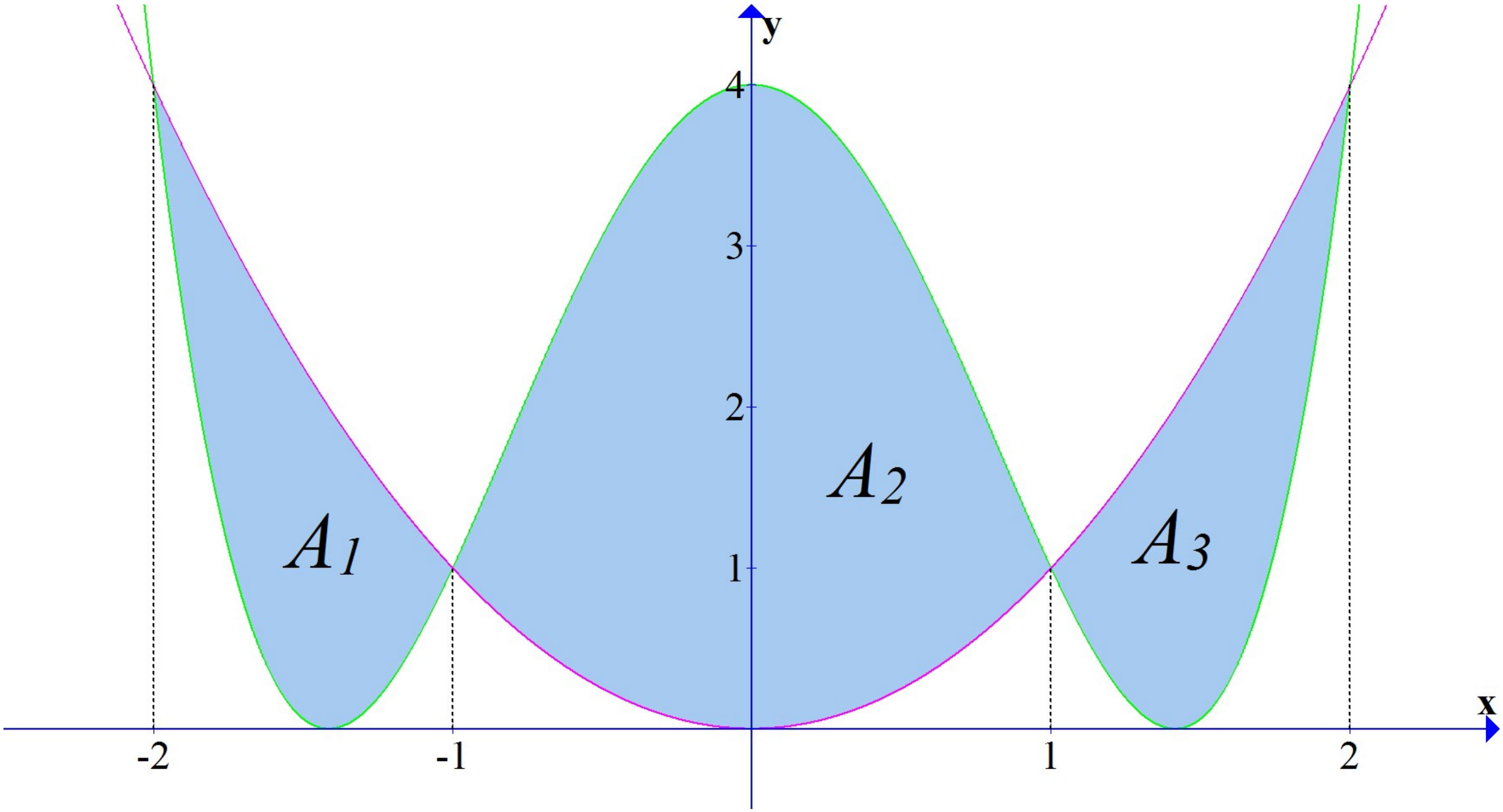
Example 5 (continued)

But the question is, where do the two curves intersect?

$$x^4 - 4x^2 + 4 = x^2$$
$$x^4 - 5x^2 + 4 = 0$$
$$(x^2 - 4)(x^2 - 1) = 0$$
$$(x - 2)(x + 2)(x - 1)(x + 1) = 0$$

So the curves intersect when $x = 2$, $x = -2$, $x = 1$ and $x = -1$.

Example 5 (continued)



Example 5 (continued)

$$A_1 = \int_{-2}^{-1} [x^2 - (x^4 - 4x^2 + 4)]\, dx$$

$$= \int_{-2}^{-1} [-x^4 + 5x^2 - 4]\, dx$$

$$= \left(-\frac{1}{5}x^5 + 5 \cdot \frac{1}{3}x^3 - 4x \right) \Bigg|_{-2}^{-1}$$

$$= \left(-\frac{1}{5}(-1)^5 + 5 \cdot \frac{1}{3}(-1)^3 - 4(-1) \right)$$

$$- \left(-\frac{1}{5}(-2)^5 + 5 \cdot \frac{1}{3}(-2)^3 - 4(-2) \right)$$

$$= \left(\frac{1}{5} - \frac{5}{3} + 4 \right) - \left(\frac{32}{5} - \frac{40}{3} + 8 \right)$$

Example 5 (continued)

$$A_2 = \int_{-1}^{1} [(x^4 - 4x^2 + 4) - x^2]\,dx = \int_{-1}^{1} [x^4 - 5x^2 + 4]\,dx$$

$$= \left(\frac{1}{5}x^5 - 5 \cdot \frac{1}{3}x^3 + 4x\right)\Bigg|_{-1}^{1}$$

$$= \left(\frac{1}{5} \cdot 1^5 - 5 \cdot \frac{1}{3} \cdot 1^3 + 4 \cdot 1\right) - \left(\frac{1}{5}(-1)^5 - 5 \cdot \frac{1}{3}(-1)^3 + 4(-1)\right)$$

$$= \left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)$$

$$A_3 = \int_{1}^{2} [x^2 - (x^4 - 4x^2 + 4)]\,dx = \int_{1}^{2} [-x^4 + 5x^2 - 4]\,dx$$

$$= \left(-\frac{1}{5}x^5 + 5 \cdot \frac{1}{3}x^3 - 4x\right)\Bigg|_{1}^{2}$$

$$= \left(-\frac{1}{5} \cdot 2^5 + 5 \cdot \frac{1}{3} \cdot 2^3 - 4 \cdot 2\right) - \left(-\frac{1}{5} \cdot 1^5 + 5 \cdot \frac{1}{3} \cdot 1^3 - 4 \cdot 1\right)$$

$$= \left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)$$

Example 5 (continued)

$$
\begin{aligned}
A &= A_1 + A_2 + A_3 \\
&= \left[\left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(\frac{32}{5} - \frac{40}{3} + 8\right)\right] \\
&+ \left[\left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] \\
&+ \left[\left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] \\
&= -\frac{60}{5} + \frac{60}{3} = -12 + 20 = 8
\end{aligned}
$$

3 OUT OF 2 PEOPLE HAVE TROUBLE WITH FRACTIONS