Substitution and Area Between Curves

Part 1

Theorem

If $u = g(x)$ and $\frac{du}{dx} = g'(x)$ is continuous on $[a, b]$ and if f is continuous on $R(g)$ (= the range of g), then

$$\int_a^b \left(f(u) \frac{du}{dx} \right) dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Theorem Rough Proof

Let $F'(x) = f(x)$. Then

$$\frac{d}{dx}\left[F\big(g(x)\big)\right] = F'\big(g(x)\big) \cdot g'(x)$$

$$= F'(u) \cdot \frac{du}{dx}$$

$$= f(u)\frac{du}{dx}$$

So the left hand side of the equation on the Theorem becomes:

Theorem Rough Proof (continued)

$$\int_a^b \left(f(u) \frac{du}{dx} \right) dx = \int_a^b \frac{d}{dx} \left[F\big(g(x)\big) \right] dx$$

$$= F\big(g(x)\big)\Big|_a^b = F\big(g(b)\big) - F\big(g(a)\big)$$

$$= F(u)\Big|_{g(a)}^{g(b)}$$

$$= \int_{g(a)}^{g(b)} f(u)\, du,$$

proving our theorem.

Example 1

Evaluate

$$\int_0^2 2x(x^2+1)^3\,dx$$

<u>Solution</u>:

We will use substitution to solve this.

$$u = x^2 + 1$$

$$du = 2xdx$$

Example 1 (continued)

When we make the substitution, we are no longer in "x" land – we have moved to "u" land, so we need to change every part of the original integral to be in terms of u – including the limits of integration.

$$u = x^2 + 1$$

$$du = 2xdx$$

Upper limit of integration:

$$x = 2 \Rightarrow u = 2^2 + 1 = 5$$

Lower limit of integration:

$$x = 0 \Rightarrow u = 0^2 + 1 = 1$$

Example 1 (continued)

$$\int_0^2 2x(x^2+1)^3\,dx = \int_0^2 (x^2+1)^3 \cdot 2x\,dx$$

$$= \int_1^5 u^3\,du = \frac{1}{4}u^4\Big|_1^5$$

$$= \frac{1}{4}\cdot 5^4 - \frac{1}{4}\cdot 1^4 = 156$$

Notice that since we changed the limits of integration, we did not need to replace u with its equivalent in x.

Example 2

Evaluate

$$\int_0^{\pi/4} \cos(\pi - x)\, dx$$

<u>Solution</u>:

$$u = \pi - x$$
$$du = -dx \Rightarrow -du = dx$$
$$x = \frac{\pi}{4} \Rightarrow u = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$
$$x = 0 \Rightarrow u = \pi - 0 = \pi$$

Example 2 (continued)

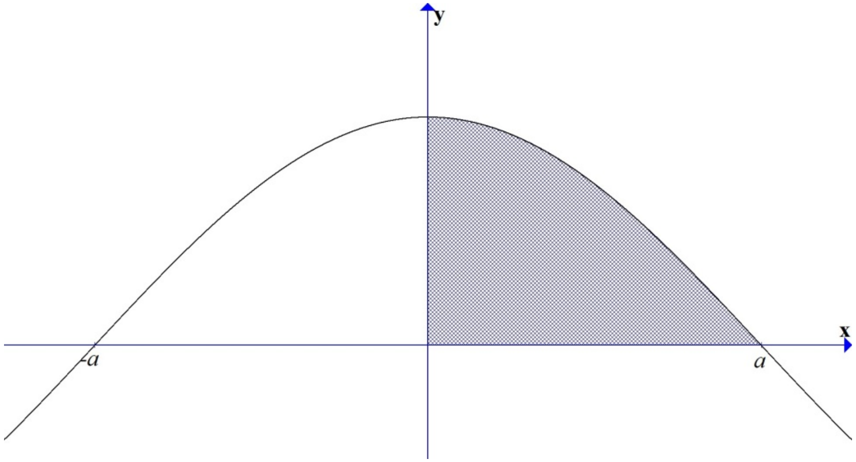
$$\int_0^{\pi/4} \cos(\pi - x)\, dx = \int_{\pi}^{3\pi/4} \cos(u)\,(-\,du\,)$$

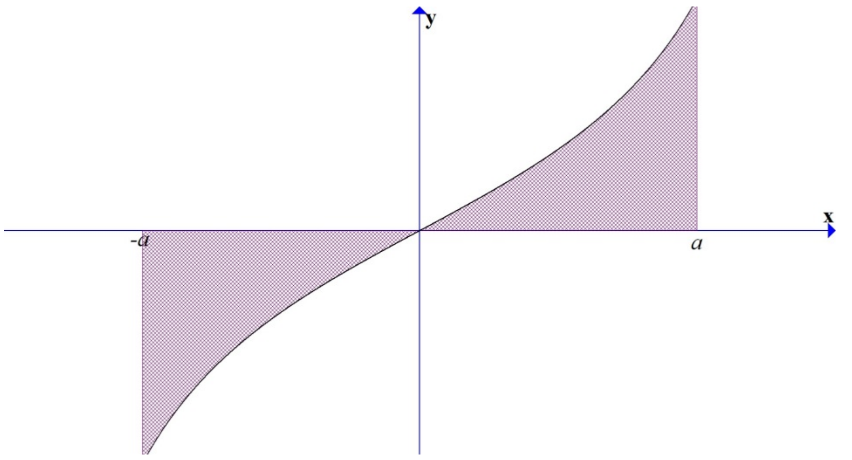
$$= \int_{3\pi/4}^{\pi} \cos(u)\, du = \sin(u)\Big|_{3\pi/4}^{\pi}$$

$$= \sin(\pi) - \sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

Theorem

- If f is even, then
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$
- If f is odd, then
$$\int_{-a}^{a} f(x)\,dx = 0$$





Example 3

$$\int_{-\pi/4}^{\pi/4} \cos(x)\, dx$$

$$= 2\int_0^{\pi/4} \cos(x)\, dx = 2\sin(x)\big|_0^{\pi/4}$$

$$= 2\sin\left(\frac{\pi}{4}\right) - 2\sin(0) = \sqrt{2}$$

$$\int_{-\pi/4}^{\pi/4} \sin(x)\, dx$$

$$= 0$$

Home On the Range