The Definite Integral

Part 1

Riemann Sums

Let $f(x)$ be any function defined on a closed interval $[a, b]$ and

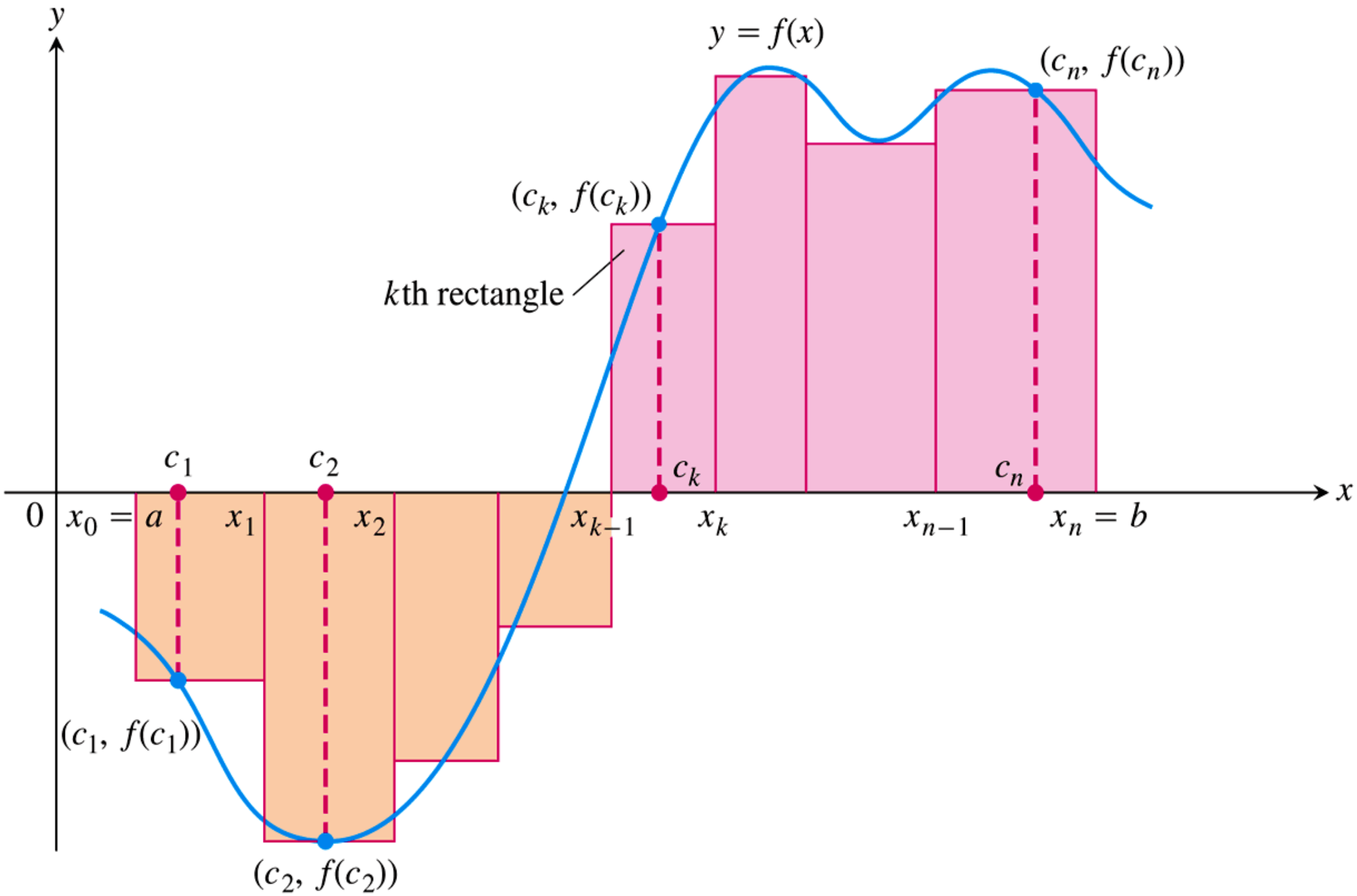
$P = \{x_0, x_1, x_2, x_3, \cdots, x_{n-1}, x_n\}$ be a partition of the interval. Then the **Riemann sum** is

$$S_P = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \cdots + f(c_n)\Delta x_n$$

$$= \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

If we increase n in such a way that $\|P\| \to 0$, then the width of every rectangle tends to zero.

Riemann Sums



Area Under a Curve

If the function $f(x)$ is continuous and non-negative on $[a, b]$, then the **area under the curve $\boldsymbol{y = f(x)}$ over the interval $[\boldsymbol{a}, \boldsymbol{b}]$** is

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

where $c_1, c_2, c_3, \cdots, c_n$ are arbitrary points in successive subintervals.

Definite Integral

If the function $f(x)$ is defined on $[a, b]$, then the **definite integral of $\boldsymbol{f}$ from $\boldsymbol{a}$ to $\boldsymbol{b}$** is

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

provided the limit exists.

$$\int_a^b f(x)\,dx$$

- $\int$ is the **integral sign**
- dx is the **variable of integration**
- a is the **lower limit of integration**
- b is the **upper limit of integration**
- $f(x)$ is the **integrand**

Example 1

Express

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} 4c_k(1 - 3c_k)\Delta x_k$$

where P is a partition of [-3,3] as a definite integral.

Example 1 (continued)

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} 4c_k(1-3c_k)\Delta x_k$$

Solution:

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k$$

We have

$$f(c_k) = 4c_k(1-3c_k) \text{ and } [a,b] = [-3,3]$$

Therefore

$$f(x) = 4x(1-3x)$$

Answer:

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} 4c_k(1-3c_k)\Delta x_k = \int_{-3}^{3} 4x(1-3x)\,dx$$

Areas Under a Curve, Riemann Sums, and Definite Integrals

If the function $f(x)$ is continuous and non-negative on $[a, b]$, then

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \int_a^b f(x)\, dx$$

Example 2

Graph the integrand and use areas to evaluate

$$\int_{-5}^{5} \sqrt{25 - x^2}\, dx.$$

<u>Solution</u>:

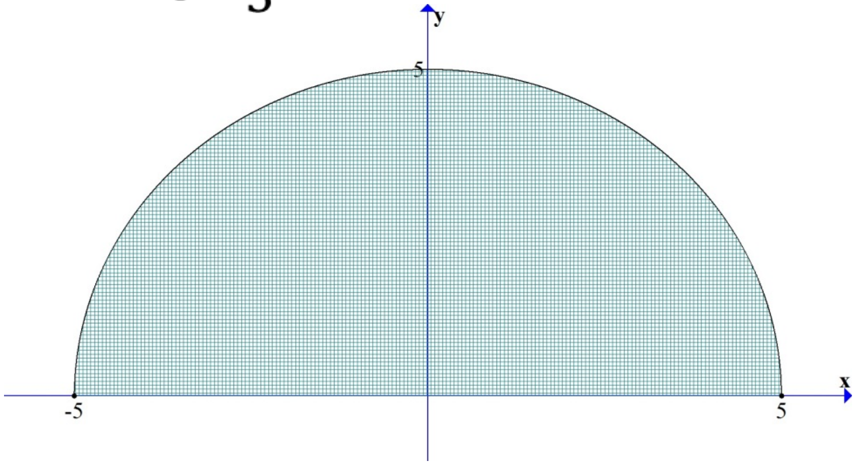
In general, $y = \sqrt{r^2 - x^2}$ is the upper semi-circle of radius r centered at the origin.

Example 2 (continued)

$$r = 5$$

Since the area of a circle of radius r is πr^2, the area of a semi-circle is $\frac{1}{2}\pi r^2$.

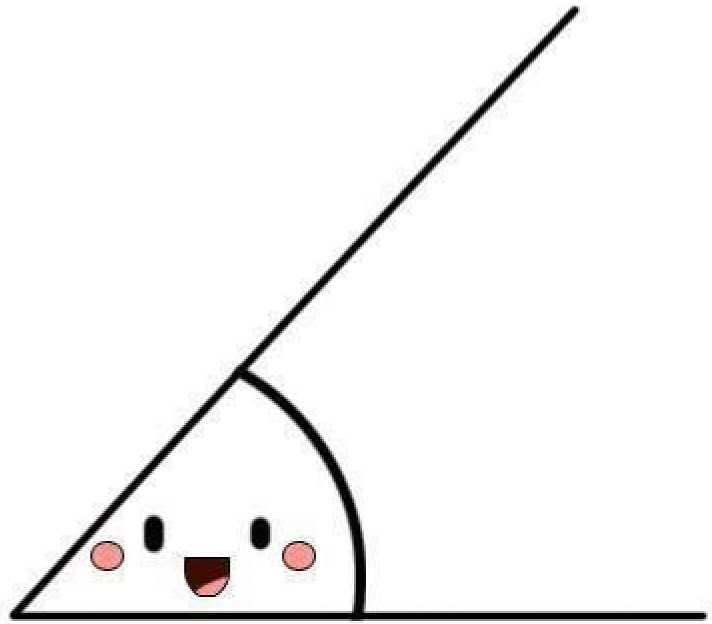
$$\int_{-5}^{5} \sqrt{25 - x^2}\, dx$$



<u>Answer</u>:

$$\int_{-5}^{5} \sqrt{25 - x^2}\, dx = \frac{1}{2}\pi(5)^2$$

$$= \frac{25\pi}{2}$$



Acute angle