Areas and Lengths in Polar Coordinates

Part 2: Lengths

Arc Length of a Polar Curve

If a curve has the polar equation $r = f(\theta)$, where $f'(\theta)$ is continuous for $\alpha \leq \theta \leq \beta$,

then its **arc length** L from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

or equivalently

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Arc Length of a Polar Curve (Rough Proof)

<u>Very Rough Proof</u>:

In rectangular coordinates:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy/d\theta}{dx/d\theta}\right)^2}\, dx$$

$$= \int_a^b \sqrt{\left(\frac{dx/d\theta}{dx/d\theta}\right)^2 + \left(\frac{dy/d\theta}{dx/d\theta}\right)^2}\, dx$$

$$= \int_a^b \sqrt{\frac{(dx/d\theta)^2 + (dy/d\theta)^2}{(dx/d\theta)^2}}\, dx$$

Arc Length of a Polar Curve (Rough Proof continued)

$$L = \int_a^b \frac{\sqrt{\left(-r\sin\theta + \cos\theta\frac{dr}{d\theta}\right)^2 + \left(r\cos\theta + \sin\theta\frac{dr}{d\theta}\right)^2}}{dx/d\theta}\,dx$$

$$= \int_a^b \frac{\sqrt{r^2\sin^2\theta - 2r\sin\theta\cos\theta\frac{dr}{d\theta} + \cos^2\theta\left(\frac{dr}{d\theta}\right)^2 + r^2\cos^2\theta + 2r\sin\theta\cos\theta\frac{dr}{d\theta} + \sin^2\theta\left(\frac{dr}{d\theta}\right)^2}}{dx/d\theta}\,dx$$

$$= \int_a^b \frac{\sqrt{r^2\sin^2\theta + r^2\cos^2\theta + \cos^2\theta\left(\frac{dr}{d\theta}\right)^2 + \sin^2\theta\left(\frac{dr}{d\theta}\right)^2}}{dx/d\theta}\,dx$$

Arc Length of a Polar Curve (Rough Proof continued)

$$L = \int_a^b \frac{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}{dx/d\theta}\, dx$$

Making the change:

$$a = r\cos\alpha \Rightarrow \theta = \cos^{-1}\frac{a}{r} \; (= \alpha)$$

$$b = r\cos\beta \Rightarrow \theta = \cos^{-1}\frac{b}{r} \; (= \beta)$$

$$dx = \frac{dx}{d\theta}\, d\theta \Rightarrow \frac{dx}{dx/d\theta} = d\theta$$

we get:

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 1

Find the arc length of the spiral $r = e^{\theta}$ between $\theta = 0$ and $\theta = 1$.

<u>Solution</u>:

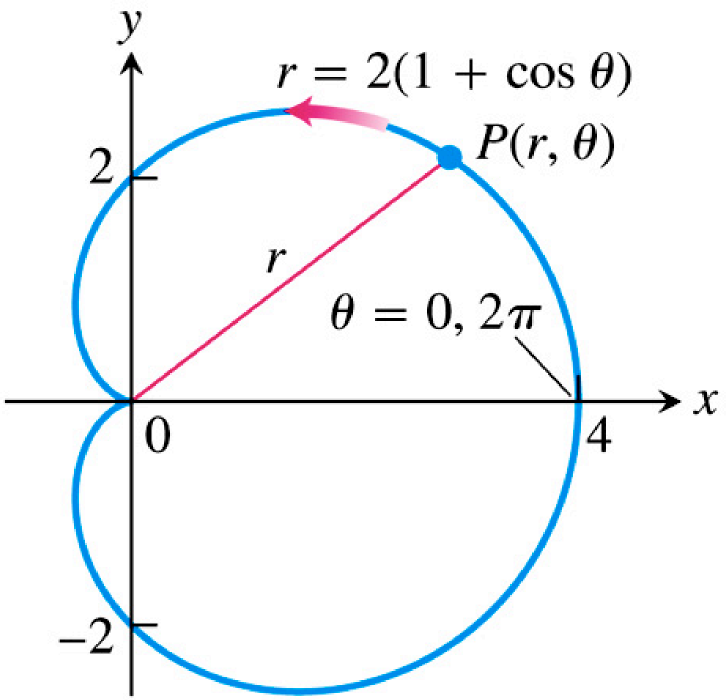
$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{1} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta$$

$$= \int_{0}^{1} \sqrt{2}e^{\theta}\, d\theta = \sqrt{2}e^{\theta}\Big|_{0}^{1} = \sqrt{2}(e-1).$$

Example 2

Find the total arc length of the cardioid

$$r = 2(1 + \cos\theta).$$



Example 2 (continued)

Solution:

The cardioid is traced out once as θ varies from $\theta = 0$ to $\theta = 2\pi$.

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{\big(2(1+\cos\theta)\big)^2 + (-2\sin\theta)^2}\, d\theta$$

Example 2 (continued)

$$L = \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta = 2\sqrt{2}\int_0^{2\pi} \sqrt{1 + \cos\theta}\, d\theta$$

$$= 2\sqrt{2}\int_0^{2\pi} \sqrt{2\cos^2\frac{\theta}{2}\, d\theta} = 4\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta$$

Example 2 (continued)

Since

$$\cos\frac{\theta}{2} \geq 0 \text{ when } 0 \leq \theta \leq \pi$$

and

$$\cos\frac{\theta}{2} \leq 0 \text{ when } \pi \leq \theta \leq 2\pi$$

we get

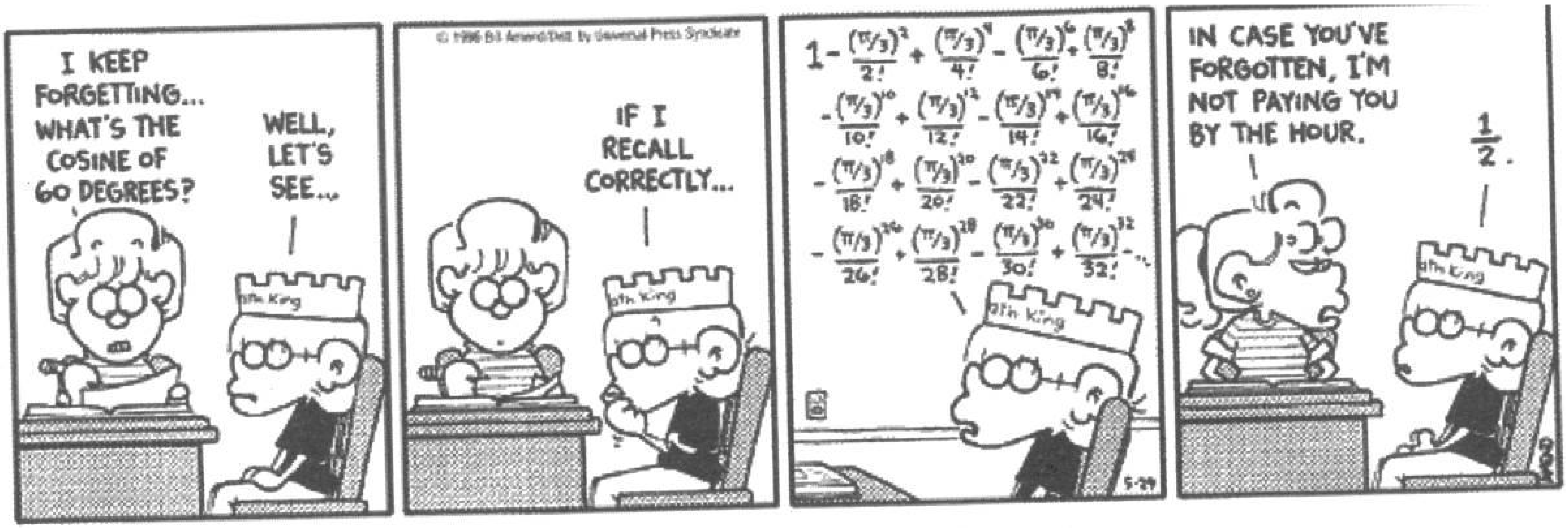
$$L = 4\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta = 4\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta - 4\int_{\pi}^{2\pi} \cos\frac{\theta}{2}\, d\theta$$

Example 2 (continued)

$$L = 4\int_0^{\pi} \cos\frac{\theta}{2}\,d\theta - 4\int_{\pi}^{2\pi} \cos\frac{\theta}{2}\,d\theta$$

$$= 8\sin\frac{\theta}{2}\bigg|_0^{\pi} - 8\sin\frac{\theta}{2}\bigg|_{\pi}^{2\pi}$$

$$= 16$$



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