**Problem 7.1**

The amplitudes $A_m$ take the values

$$A_m = (2m - 1 - M)\frac{d}{2}, \qquad m = 1, \dots M$$

Hence, the average energy is

$$\begin{aligned}
\mathcal{E}_{av} &= \frac{1}{M}\sum_{m=1}^{M} s_m^2 = \frac{d^2}{4M}\mathcal{E}_g \sum_{m=1}^{M}(2m-1-M)^2 \\
&= \frac{d^2}{4M}\mathcal{E}_g \sum_{m=1}^{M}[4m^2 + (M+1)^2 - 4m(M+1)] \\
&= \frac{d^2}{4M}\mathcal{E}_g \left(4\sum_{m=1}^{M} m^2 + M(M+1)^2 - 4(M+1)\sum_{m=1}^{M} m\right) \\
&= \frac{d^2}{4M}\mathcal{E}_g \left(4\frac{M(M+1)(2M+1)}{6} + M(M+1)^2 - 4(M+1)\frac{M(M+1)}{2}\right) \\
&= \frac{M^2-1}{3}\frac{d^2}{4}\mathcal{E}_g
\end{aligned}$$

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**Problem 7.2**

The correlation coefficient between the $m^{\text{th}}$ and the $n^{\text{th}}$ signal points is

$$\gamma_{mn} = \frac{\mathbf{s}_m \cdot \mathbf{s}_n}{|\mathbf{s}_m||\mathbf{s}_n|}$$

where $\mathbf{s}_m = (s_{m1}, s_{m2}, \dots, s_{mN})$ and $s_{mj} = \pm\sqrt{\frac{\mathcal{E}_s}{N}}$. Two adjacent signal points differ in only one coordinate, for which $s_{mk}$ and $s_{nk}$ have opposite signs. Hence,

$$\begin{aligned}
\mathbf{s}_m \cdot \mathbf{s}_n &= \sum_{j=1}^{N} s_{mj}s_{nj} = \sum_{j\neq k} s_{mj}s_{nj} + s_{mk}s_{nk} \\
&= (N-1)\frac{\mathcal{E}_s}{N} - \frac{\mathcal{E}_s}{N} = \frac{N-2}{N}\mathcal{E}_s
\end{aligned}$$

Furthermore, $|\mathbf{s}_m| = |\mathbf{s}_n| = (\mathcal{E}_s)^{\frac{1}{2}}$ so that

$$\gamma_{mn} = \frac{N-2}{N}$$

The Euclidean distance between the two adjacent signal points is

$$d = \sqrt{|\mathbf{s}_m - \mathbf{s}_n|^2} = \sqrt{\left|\pm 2\sqrt{\mathcal{E}_s/N}\right|^2} = \sqrt{4\frac{\mathcal{E}_s}{N}} = 2\sqrt{\frac{\mathcal{E}_s}{N}}$$

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**Problem 7.9**

**a)** The received signal may be expressed as

$$r(t) = \begin{cases} n(t) & \text{if } s_0(t) \text{ was transmitted} \\ A + n(t) & \text{if } s_1(t) \text{ was transmitted} \end{cases}$$

Assuming that $s(t)$ has unit energy, then the sampled outputs of the crosscorrelators are

$$r = s_m + n, \qquad m = 0, 1$$

where $s_0 = 0$, $s_1 = A\sqrt{T}$ and the noise term $n$ is a zero-mean Gaussian random variable with variance

$$\begin{aligned}
\sigma_n^2 &= E\left[\frac{1}{\sqrt{T}}\int_0^T n(t)dt \frac{1}{\sqrt{T}}\int_0^T n(\tau)d\tau\right] \\
&= \frac{1}{T}\int_0^T\int_0^T E\left[n(t)n(\tau)\right]dtd\tau \\
&= \frac{N_0}{2T}\int_0^T\int_0^T \delta(t-\tau)dtd\tau = \frac{N_0}{2}
\end{aligned}$$

The probability density function for the sampled output is

$$\begin{aligned}
f(r|s_0) &= \frac{1}{\sqrt{\pi N_0}}e^{-\frac{r^2}{N_0}} \\
f(r|s_1) &= \frac{1}{\sqrt{\pi N_0}}e^{-\frac{(r-A\sqrt{T})^2}{N_0}}
\end{aligned}$$

Since the signals are equally probable, the optimal detector decides in favor of $s_0$ if

$$\mathrm{PM}(\mathbf{r}, \mathbf{s}_0) = f(r|s_0) > f(r|s_1) = \mathrm{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of $s_1$. The decision rule may be expressed as

$$\frac{\mathrm{PM}(\mathbf{r}, \mathbf{s}_0)}{\mathrm{PM}(\mathbf{r}, \mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

or equivalently

$$r \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1}{2}A\sqrt{T}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.
**b)** The average probability of error is

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} f(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} f(r|s_1)dr \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{r^2}{N_0}}dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{(r-A\sqrt{T})^2}{N_0}}dr \\
&= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{\mathrm{SNR}}\right]
\end{aligned}$$

where

$$\text{SNR} = \frac{\frac{1}{2}A^2T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

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**Problem 7.10**

Since the rate of transmission is $R = 10^5$ bits/sec, the bit interval $T_b$ is $10^{-5}$ sec. The probability of error in a binary PAM system is

$$P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

where the bit energy is $\mathcal{E}_b = A^2T_b$. With $P(e) = P_2 = 10^{-6}$, we obtain

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 4.75 \Longrightarrow \mathcal{E}_b = \frac{4.75^2 N_0}{2} = 0.112813$$

Thus

$$A^2T_b = 0.112813 \Longrightarrow A = \sqrt{0.112813 \times 10^5} = 106.21$$

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**Problem 7.11**

**a)** For a binary PAM system for which the two signals have unequal probability, the optimum detector is

$$r \underset{s_2}{\overset{s_1}{\gtrless}} \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln\frac{1-p}{p} = \eta$$

The average probability of error is

$$\begin{aligned}
P(e) &= P(e|s_1)P(s_1) + P(e|s_2)P(s_2) \\
&= pP(e|s_1) + (1-p)P(e|s_2) \\
&= p\int_{-\infty}^{\eta} f(r|s_1)dr + (1-p)\int_{\eta}^{\infty} f(r|s_1)dr \\
&= p\int_{-\infty}^{\eta} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{\mathcal{E}_b})^2}{N_0}} dr + (1-p)\int_{\eta}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r+\sqrt{\mathcal{E}_b})^2}{N_0}} dr \\
&= p\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\eta_1} e^{-\frac{x^2}{2}}dx + (1-p)\frac{1}{\sqrt{2\pi}}\int_{\eta_2}^{\infty} e^{-\frac{x^2}{2}}dx
\end{aligned}$$

where

$$\eta_1 = -\sqrt{\frac{2\mathcal{E}_b}{N_0}} + \eta\sqrt{\frac{2}{N_0}} \qquad \eta_2 = \sqrt{\frac{2\mathcal{E}_b}{N_0}} + \eta\sqrt{\frac{2}{N_0}}$$

Thus,

$$P(e) = pQ\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \eta\sqrt{\frac{2}{N_0}}\right] + (1-p)Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} + \eta\sqrt{\frac{2}{N_0}}\right]$$

**b)** If $p = 0.3$ and $\frac{\mathcal{E}_b}{N_0} = 10$, then

$$\begin{aligned}
P(e) &= 0.3Q[4.3774] + 0.7Q[4.5668] = 0.3 \times 6.01 \times 10^{-6} + 0.7 \times 2.48 \times 10^{-6} \\
&= 3.539 \times 10^{-6}
\end{aligned}$$

If the symbols are equiprobable, then

$$P(e) = Q[\sqrt{\frac{2\mathcal{E}_b}{N_0}}] = Q[\sqrt{20}] = 3.88 \times 10^{-6}$$

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**Problem 7.12**

**a)** The optimum threshold is given by

$$\eta = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1-p}{p} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$$

**b)** The average probability of error is ($\eta = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$)

$$\begin{aligned}
P(e) &= p(a_m = -1) \int_\eta^\infty \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{\mathcal{E}_b})^2/N_0} dr \\
&\quad + p(a_m = 1) \int_{-\infty}^\eta \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0} dr \\
&= \frac{2}{3} Q\left[\frac{\eta + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}}\right] + \frac{1}{3} Q\left[\frac{\sqrt{\mathcal{E}_b} - \eta}{\sqrt{N_0/2}}\right] \\
&= \frac{2}{3} Q\left[\frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4}\right]
\end{aligned}$$